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Efficiency and Endogenous Fertility*

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Abstract

This paper explores the properties of the notions of $\mathcal{A}-$efficiency and $\mathcal{P}-$efficiency, proposed by Golosov, Jones and Tertilt (*Econometrica*, 2007), to evaluate allocations in a general overlapping generations setting in which fertility choices are endogenously selected from a *continuum* and any two agents of the same generation are identical. First, we show that the properties of $\mathcal{A}-$efficient allocations vary depending on the criterion used to identify potential agents. If one identifies potential agents by their position in their siblings' birth order –as Golosov, Jones and Tertilt do–, then $\mathcal{A}-$efficiency requires that a positive measure of agents use most of their endowment to maximize the utility of the dynasty head, which, in environments with finite horizon altruism, implies that some agents –the youngest in every family– obtain an arbitrary low income to finance their own consumption and fertility plans. If potential agents are identified by the dates in which they may be born, then $\mathcal{A}-$efficiency reduces to dynastic maximization, which, in environments with finite horizon altruism, drives the economy to a collapse in finite time. To deal with situations –like those arising in economies with finite horizon altruism–, in which $\mathcal{A}-$efficiency may be in conflict with individual rights, we propose to evaluate the efficiency of a given allocation with a particular class of specifications of $\mathcal{P}-$efficiency, for which the utility attributed to the unborn depends on the utility obtained by their living siblings. Under certain concavity assumptions on value functions, we also characterize every symmetric, $\mathcal{P}$-efficient allocation as a Millian efficient allocation, that is, as a symmetric allocation that is not $\mathcal{A}-$dominated by any other symmetric allocation.

Key words: Efficiency, Optimal Population, Endogenous Fertility, $\mathcal{A}-$efficiency, $\mathcal{P}-$efficiency, Millian efficiency.

JEL: D91, H21, H5, E62, J13

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1 Introduction

The most commonly used optimality notion in normative economic analysis is that of Pareto efficiency. This notion of efficiency relies on the well-known Pareto criterion to compare social alternatives, a criterion that allows one to construct a *partial ordering* on a set of alternatives from the *complete preference orderings* (defined on this set) of a *fixed* group of agents. An efficient allocation can be described as a maximal element of the partial ordering induced by the Pareto criterion on the set of feasible allocations.

When fertility decisions are endogenous, one can still use the Pareto criterion to rank feasible allocations using the *partial orderings* of all potential agents, represented by the utility functions of the *living* agents. However, any two allocations with different fertility choices cannot be ranked, since there is no way to know whether or not an agent who lives in one allocation a but not in other allocation a' is better off in the latter than he is in the former. To avoid this problem and preserve the partial ordering induced by the Pareto criterion, one needs to extend it to compare also allocations with different fertility choices.

Although the issue seems to concern policymakers everywhere, the theoretical foundations of many proposals to alter fertility rates are rather weak. Most of the literature simply identifies *optimal allocations* with the solutions to alternative social welfare maximization problems, referred to as *Millian* or *Benthamite,* according to whether or not the welfare weight given to a generation in the social welfare function depends on the size of that generation.[1] But this approach does not take into account the fact that the Pareto criterion is not directly applicable to environments in which the set of agents is endogenous. Besides, a social welfare maximization problem identifies a unique "optimal" allocation. As pointed out by Golosov, Jones and Tertilt (2007, p.1041), "such criteria (...) are very different in spirit from an efficiency concept that usually contains a larger number of allocations."

Unlike this literature, Michel and Wigniolle (2007), and Conde-Ruiz *et al.* (2010) have provided normative principles to evaluate population policies in the context of an overlapping generations framework without altruism, in which the set of feasible fertility choices is continuous. These papers restrict their analysis to *symmetric* allocations, that is, to allocations in which any two agents of the same generation obtain the same consumption bundle, and focus on an extension of the Pareto criterion –which is referred to as *Representative Consumer dominance* by Michel *et al.* and as $\mathcal{A}-$*dominance* by Conde-Ruiz *et al.*– that ranks any two allocations of different population size by comparing exclusively the welfare profiles of those agents who are alive in the two allocations. Given the symmetry restriction on the set of allocations that are comparable using the $\mathcal{A}-$dominance criterion, a feasible symmetric allocation is said to be efficient (or, using the term proposed by Conde-Ruiz *et al.*, *Millian efficient*) if there does not exist an alternative feasible, symmetric allocation that provides all members of a generation with higher utility without decreasing the utility obtained by any other generation. The name "Millian" refers to the fact that it is a notion of efficiency that generalizes the notion of Millian optimality mentioned above.

But the restriction to symmetric allocations in environments without altruism limits the scope of these papers. To fill this gap, Golosov, Jones and Tertilt (2007) have proposed two alternative extensions of the Pareto criterion in the context of a general model in which the

[1] For a discussion on the different notions of optimality arising in settings with endogenous fertility, see e.g., Razin and Sadka (1995, Ch.5).

set of fertility choices is discrete. The first one is the $\mathcal{A}-$dominance criterion mentioned above, without restricting welfare comparisons to symmetric allocations. The second, referred to as the $\mathcal{P}-$dominance criterion, is constructed from a preliminary assumption on the utility level obtained by potential non-born agents. These two extensions of the notion of Pareto dominance give rise to two notions of efficiency, respectively referred to as $\mathcal{A}-$efficiency and $\mathcal{P}-$efficiency, to evaluate allocations in environments in which fertility decisions are endogenous. Golosov, Jones and Tertilt (henceforth GJT) provide partial characterizations of the two notions of efficiency as the solutions to welfare maximization problems, and prove that, under relatively mild assumptions, $\mathcal{A}-$efficient allocations are either $\mathcal{P}-$efficient or are arbitrarily close to allocations that are also $\mathcal{P}-$efficient (see GJT 2007, Sec.4.3 Result 3). Thus, the $\mathcal{P}-$efficiency of $\mathcal{A}-$efficient allocations is robust to different specifications of the utility levels attributed to the unborn. The authors also extend their analysis to a setting in which fertility choices are selected from a *continuum* and agents exhibit *dynastic* altruism *à la* Barro and Becker (1989). In this context, GJT explore the properties of a notion of equilibrium which results from the combination of the notion of competitive equilibrium and the notion of subgame perfect equilibrium of a transfer game played within families, and provide a version of the First Welfare Theorem by showing that said equilibrium is both $\mathcal{A}-$efficient and $\mathcal{P}-$efficient (see GJT 2007, Th.2).

A seemingly irrelevant feature of GJT's work deserves some discussion. The authors, for whom fertility choices are represented by the number of children that parents decide to bear during a child-rearing period, identify the agents who may be potentially alive in an economy by the identities of their parents and their position in their siblings' birth order. For example, agent $i = (1, 2)$ identifies the second child of the first child of the dynasty head. There are, however, other possibilities. If people are born at different points in time (as GTJ implicitly assume when they identify potential agents by their position in their siblings' birth order) and parents are able to choose not only the number of children they are willing to rear, but also the specific points in time –within the child-rearing period– at which they want to give birth to these children, we may identify potential agents by their parents' identities and the points in time at which they may be born. Under this criterion, for example, agent $i = (1, 2)$ identifies the child born at $t_2 = 2$ from the descendant of the dynasty head born at $t_1 = 1$. One might think that, if none of the agents potentially alive in an economy cares about the specific dates at which they give birth to their children, then the criterion used to distinguish among potential agents should be irrelevant. However, from the point of the $\mathcal{A}-$dominance criterion, it is not. As we shall explain throughout the paper, the eldest children in a family will be alive in all allocations in which their parents have children, and their preferences must be taken into account if one evaluates an increase in their family size using the $\mathcal{A}-$dominance criterion and potential agents are identified by their positions in the birth order. In contrast, when potential agents are identified by their birth dates, the eldest children in a family might not even be born in many alternative allocations in which their parents have more children. Therefore, their preferences need not be taken into account if one evaluates an increase in their family size using the $\mathcal{A}-$dominance criterion.

In this paper, we study the properties of the different notions of efficiency in a general, overlapping generations setting in which *a*) the number of children within each family is endogenously selected from a bounded interval in the positive real line, the child-rearing period, that represents also the *continuum* of time instants at which children may be born;

b) the agents are able to decide –possibly, with constraints– not only the number of children they bear, but also the set of specific points in time at which they want to give birth to their children; and *c*) any two living agents of the same generation have the same labour endowment and the same preferences, which depend on their own consumption of a homogeneous good, on the number of children they decide to bear and on the welfare obtained by their descendants. Thus, our setting generalizes GJT's continuous model of fertility choices in two respects. First, it covers, as particular cases, a wide range of positive models of fertility choice, including not only Barro and Becker's, but also others in which altruism lies between the two polar representations (no altruism and dynastic altruism) considered mainly in the literature. Second, potential agents may be identified with the two criteria described above, that is, by the agents' positions in their siblings' birth order or by their birth dates. In what follows, we shall refer to the former criterion as the *Birth-Order* criterion and to the latter as the *Birth-Date* criterion. The results of the paper can be gathered in two blocks.

$\mathcal{A}-$efficiency. As mentioned above, the properties of $\mathcal{A}-$efficient allocations depend on the criteria to confer the agents potentially alive in an economy their identity.

i) We focus first on the *Birth-Order criterion* –the one used by GJT– by exploring the properties of $\mathcal{A}-$efficiency in a setting in which fertility choices are constrained in such a way that the two criteria to distinguish among potential agents coincide. In this case, for every $\mathcal{A}-$efficient allocation, the youngest individuals in every family must devote most of their entire income to maximize not their own utility, but the utility of their parents and, hence, of the dynasty head (Theorem 1). When altruism of the agents is of the finite horizon type, Theorem 1 implies that *every* $\mathcal{A}-$efficient allocation is characterized by providing a positive measure of agents of every generation born after period $t = 2$ with almost no resources to finance their own consumption (Corollary 1).

Indeed, starting from an allocation a in which none of the agents uses his/her income to maximize the utility of the dynasty head, it is always possible to find another allocation a' with more individuals that makes all people living under both a and a' better off than they were under a, irrespectively of how potential agents are identified. Welfare improvements of this type (in the sense given by the $\mathcal{A}-$dominance criterion) can be achieved by enforcing every newcomer –that is, every individual living under a' who is not born under a– to use their endowment to maximize their parents' utilities. Newcomers can provide their parents with at least the same utility as those already living in a. Moreover, since marginal agents already living in a were not maximizing their parents' utilities, newcomers require fewer resources in order to achieve this objective. Finally, even though newcomers would rather prefer to obtain the same consumption bundles than those already living in a, this involves no welfare losses from the point of view of the $\mathcal{A}-$dominance criterion.

ii) When potential agents are identified with the *Birth-Date criterion*, the set of $\mathcal{A}-$efficient allocations shrinks. In Theorem 2 , we show that, when fertility choices are unrestricted and the child-rearing period is long enough, every allocation arising as $\mathcal{A}-$efficient with this criterion can be characterized as a dynastic optimum. This result can be extended to discrete settings and, therefore, contrasts with GJT's examples showing that $\mathcal{A}-$efficiency differs from dynastic maximization. If, in addition, altruism of the agents is of the finite horizon type, Theorem 2 implies that in *every* $\mathcal{A}-$efficient allocation, the economy collapses in finite time. The argument behind Theorem 2 is simple: if some of the agents born in a

given allocation do not use their endowment to maximize the utility of the dynasty head, she can always replace these agents by a set of agents of equal measure who do behave as she wishes.

In view of the differences between the two criteria to confer the agents their identity, a question arises: which criterion should we opt for? One might think that the *Birth-Order criterion* seems more natural, because the eldest children in a family will be already alive when their younger siblings are born. However, in both GTJ's model and the class of models presented here, all descendants will be alive by the time parents decide whether or not they are willing to provide them with any resources. In addition, the objective behind extending the Pareto criterion is to evaluate decisions such as increasing the size of a family *before* potential agents are born, not while potential agents are being born nor after they are all born. Therefore, we see no reason, from a normative point of view, to identify potential agents in a family using the birth order.

Furthermore, the fact that achieving $\mathcal{A}-$efficiency always requires that some of the agents must devote their entire endowment to maximize the utility of their parents shows that there are significant differences between the notion of $\mathcal{A}-$efficiency and the notion of Pareto efficiency. These differences might introduce difficulties in the implementation of $\mathcal{A}-$efficient outcomes, not only because, in some environments, achieving $\mathcal{A}-$efficiency may leave some of the living agents with almost no resources, but also because, even in environments with dynastic altruism, $\mathcal{A}-$efficiency might be incompatible with the agents' rights to use some of their resources –for example, their labour capacities– as they wish. But, beyond these differences on the properties of efficient allocations, our analysis of $\mathcal{A}-$efficiency in different environments raises, in our view, some doubts on the very notion of $\mathcal{A}-$dominance as a criterion to aggregate individual preferences: with this criterion, increasing the population size increases social welfare as long as all (already) living agents are better off, irrespectively of the living conditions of the newborn.

Millian efficiency as robust $\mathcal{P}-$efficiency. In the paper, we also explore whether the results in *i*) or *ii*) hold for the notion of $\mathcal{P}-$efficiency proposed by GJT. Differently from these authors, for whom the utility of the unborn is a constant value $\overline{u}$, we explore the possibility that the utility attributed to the unborn is a symmetric function of the utility achieved by the agent's living siblings. By making the utility attributed to the unborn depends on the utility obtained by those alive in a given allocation, we introduce, in welfare evaluation, principles such as "An increase in the population size is not welfare improving if the newborn are worse off than any –or most, or the average– of their siblings already alive." Under certain (concavity) conditions, and independently on the criterion chosen to identify potential agents, a symmetric allocation is $\mathcal{P}-$efficient if, and only if, it is Millian efficient (see Theorem 3). Furthermore, the $\mathcal{P}-$efficiency of Millian efficient allocations holds for a wide range of specifications of the utility attributed to the unborn, that is, for a wide range of principles determining under what conditions a new life is welfare improving. By assuming that the utility attributed to an unborn agent depends exclusively on the utility obtained by the agent's living siblings, we avoid cardinal assessments in welfare comparisons. Finally, using a weaker notion of efficiency might be the best option available if altering the agents' property rights on their labour capacities is not viable.

The paper is organized as follows. In the following Section (Section 2), we anticipate and discuss our main results in the context of a simple, two period economy. In Section 3, we introduce our general model. In Section 4, we explore the properties of $\mathcal{A}-$efficient allocations in the context of the model described in the previous section. In Section 5, we explore the properties of $\mathcal{P}-$efficiency when the utility obtained by the unborn is a function of the utility obtained by their living siblings. In Section 6 we present our main conclusions and discuss several possibilities for further research.

2 Efficient fertility choices in a simple, two-period example

To illustrate the notions of $\mathcal{A}-$ and $\mathcal{P}-$efficiency and describe the definitions and the main results in the paper, we present a simple, two-period economy with two generations of agents: the *dynasty head* and her potential offspring. The economy is, therefore, analogous to those described in Examples 2 and 3 in GJT, the difference being that, in this case, the set of feasible fertility choices is continuous and represented by an interval $[0, \overline{n}]$, where $\overline{n}$ represents the maximum number of children that the dynasty head can bear. To be more precise, a fertility choice is represented by the number $n_1 \in [0, \overline{n}]$ that determines the number of children that the dynasty head chooses to rear. Following GJT, we shall initially identify potential agents by their position in the descendants' birth order.

At time $t = 0$, there are $\overline{e}_0$ units of a homogeneous good that can be used to finance the dynasty head's consumption, c_0^m, as well as capital investments, k_1^o, and child bearing activities, represented by a linear cost function $b(n_1) = bn_1$, with $b > 0$. The resource constraint at time 0 is $c_0^m + bn_1 + k_1^o \leq \overline{e}_0$, which, by writing k_1 for capital *per worker* (that is, $k_1 = k_1^o/n_1$), can be equivalently written as

$$c_0^m + n_1 (b + k_1) \leq \overline{e}_0.$$

At time $t = 1$, total output is given by $Y_1 = F(k_1^o, n_1)$. In this simple example, we focus on a linear technology $F(k_1^o, n_1) = Rk_1^o + wn$, with $R > 0$ and $w > 0$ satisfying $Rb > w$.[2] This output is used to provide with the consumption good to each of descendants of the dynasty head in case these descendants are alive. Since fertility choices are selected from a *continuum*, consumption choices made by the descendants of the dynasty head are represented by a (Lebesgue) integrable function $\mathsf{c}_1^m : [0, \overline{n}] \to \mathbb{R}_+$, where $\mathsf{c}_1^m(i)$ represents the amount of the consumption good available to the *ith* descendant. Denote by e_1 the average consumption available to the dynasty head's descendants, that is,

$$e_1 = \frac{1}{n_1} \int_0^{n_1} \mathsf{c}_1^m(i) di. \tag{1}$$

Observe that if all alive children consume the same –$\mathsf{c}_1^m(i) = \mathsf{c}_1^m(i')$ for every $(i, i') \in [0, n_1]^2$–, then $e_1 = \mathsf{c}_1^m(i)$ must be satisfied. The resource constraint, in *per capita* terms, arising at time $t = 1$ adopts the form

$$e_1 \leq F(k_1, 1) = f(k_1) = Rk_1 + w.$$

[2] If $Rb < w$, then a dynastic optimum does not exist.

By letting $k(e_1) = f^{-1}(e_1) = [e_1 - w]/R$, the resource constraint at time $t = 0$ becomes

$$c_0^m + n_1 \left[b + k(e_1)\right] \leq \overline{e}_0. \tag{2}$$

In the economy described, an allocation $\mathsf{a} = (a, \mathsf{c}_1^m)$ is a vector $a = (n_1, c_0^m, e_1) \in [0, \overline{n}] \times \mathbb{R}_+^2$ satisfying (2) and a consumption plan $\mathsf{c}_1^m : [0, \overline{n}] \to \mathbb{R}_+$ satisfying (1). An allocation a is *ex-post symmetric* (or, simply, symmetric) if it provides every descendant with the same consumption bundle and, hence, satisfies $\mathsf{c}_1^m(i) = e_1$ for every $i \in [0, n_1]$. A symmetric allocation $\mathsf{a} = (a, \mathsf{c}_1^m)$, therefore, is completely described by the vector $a = (c_0^m, n_1, e_1)$.

Each potential descendant i, if alive (*i.e.*, if $i \leq n_1$), cares monotonically about her own consumption $\mathsf{c}_1^m(i)$. The dynasty head is concerned with her own consumption, the number of descendants she decides to bear and the amount of the consumption good available to each of her descendants. Her preferences on feasible allocations are represented by a utility function of the form

$$\mathcal{U}_0(\mathsf{a}) = U\left(c_0^m, n_1, \frac{1}{n_1}\int_0^{n_1} U^D(\mathsf{c}_1^m(i))di\right),$$

where U is non-decreasing, continuously differentiable and concave and U^D is a non-decreasing, strictly concave function determining the utility obtained by the dynasty head from the consumption decisions of her descendants. In this simple example, the preferences of the dynasty head are represented by the following altruistic utility function, that adapts GJT's Examples 1 and 2 to allow for a continuous set of fertility choices:

$$\mathcal{U}_0(\mathsf{a}) = \frac{\alpha}{\sigma}[c_0^m]^\sigma + \beta\int_0^{n_1}\frac{1}{\sigma}[\mathsf{c}_1^m(i)]^\sigma di.$$

We take the same parametrization as in GJT. Specifically, $\overline{e}_0 = 100$, $b = 24$, $R = 1$, $w = 0$, $\alpha = \beta = \gamma = \sigma = 1/2$.

$\mathcal{A}-$ and Millian efficiency. An allocation $\widehat{\mathsf{a}}$ is $\mathcal{A}-$efficient if it is not $\mathcal{A}-$dominated by any other allocation; that is, there does not exist an alternative, feasible allocation a that confers higher utility to every agent living in both $\widehat{\mathsf{a}}$ and a, and strictly higher utility to some of these agents. Taking into account that the dynasty head is alive in every feasible allocation, any allocation a^* that maximizes the utility of the dynasty head must be $\mathcal{A}-$efficient.[3] Moreover, since the function U^D is strictly concave, such dynastic optimum must be symmetric. The pair (n_1^*, e_1^*) corresponding to a dynastic optimum a^* must, therefore, solve

$$\max_{(n_1, e_1)\in[0,\overline{n}]\times\mathbb{R}_+}\left\{U(\overline{e}_0 - n_1[b + k(e_1)], n_1, U^D(e_1))\right\} \equiv \mathcal{V}_0(\overline{e}_0), \tag{3}$$

which, in our parametric example, yields

$$(c_0^{m*}, n_1^*, e_1^*) = \left(24, \frac{6^{\frac{1}{2}}}{12}, 24\right).$$

Are there other $\mathcal{A}-$efficient allocations? A class of seemingly good candidates is the class of Millian efficient allocations; that is, the class of symmetric allocations that cannot

[3]This is also true when fertility choices are integers, as shown by GJT (2007, Sec.3.2).

be $\mathcal{A}-$dominated by any other symmetric allocation. A Millian efficient allocation $\widehat{\mathsf{a}}$ gives the dynasty head the maximum utility that she can obtain with a symmetric allocation if she is restricted to provide with *at least* $\widehat{e}_1$ units of resources to *each* of her descendants, *i.e.*, $e_1 \geq \widehat{e}_1$. That is, the pair $(\widehat{n}_1, \widehat{e}_1)$ corresponding to a Millian efficient allocation solves

$$\max_{(n_1,e_1)\in[0,\overline{n}]\times\mathbb{R}_+} \{U(\overline{e}_0 - n_1[b + k(e_1)], n_1, U^D(e_1)) : e_1 \geq \widehat{e}_1\}. \tag{4}$$

It is also easy to see that, if $\widehat{e}_1 \geq e_1^*$, then the constraint $e_1 \geq \widehat{e}_1$ in (4) must be binding. Therefore, Millian efficient allocations are those for which $\widehat{n}_1$ solves, for some $\widehat{e}_1 \geq e_1^*$,

$$\max_{n_1\in[0,\overline{n}]} \{U(\overline{e}_0 - n_1[b + k(\widehat{e}_1)], n_1, U^D(\widehat{e}_1))\} \equiv W_0(\overline{e}_0, \widehat{e}_1, U^D(\widehat{e}_1)). \tag{5}$$

In our parametric example, for any $\widehat{e}_1 \geq w$, the solution $(\widehat{c}_0^m, \widehat{n}_1, \widehat{e}_1) = (c_0^m(\widehat{e}_1), n_1(\widehat{e}_1), k_1(\widehat{e}_1))$ to the optimization problem in the definition of $W_0(\overline{e}_0, \widehat{e}_1, U^D(\widehat{e}_1))$ is

$$(\widehat{c}_0^m, \widehat{n}_1, \widehat{e}_1) = \left(\frac{[24 + \widehat{e}_1]^2}{16\widehat{e}_1}, \frac{100 - \left[\frac{24+\widehat{e}_1}{16\widehat{e}_1}\right]^2}{24 + \widehat{e}_1}, \widehat{e}_1 \right),$$

which implies that the fertility rate arising in the dynastic optimum is always higher than that corresponding to any other Millian efficient allocation, *i.e.* $\widehat{n}_1 \leq n_1^*$ whenever $\widehat{e}_1 \geq e_1^*$.

Clearly, starting from a symmetric allocation $\widehat{\mathsf{a}}$ for which $(\widehat{n}_1, \widehat{e}_1)$ solves the optimization problem in (4), the only way of increasing the utility of the dynasty head with a symmetric allocation is by decreasing e_1, which, taking into account that a lower income $e_1 < \widehat{e}_1$ brings with it a higher fertility level $n_1 \geq \widehat{n}_1$, would make all the descendants who where already living in the original allocation worse off.

Perhaps surprisingly, the only Millian efficient allocation that is also $\mathcal{A}-$efficient is the dynastic optimum. To see why, let $\widehat{a} = (\widehat{c}_0^m, \widehat{n}_1, \widehat{e}_1)$ be Millian efficient allocation satisfying $\widehat{e}_1 > e_1^*$. Now suppose the dynasty head is given the opportunity to choose an asymmetric allocation $\widetilde{\mathsf{a}}$ with more children than those living in $\widehat{a}$ (that is, satisfying $\widetilde{n}_1 \geq \widehat{n}_1$) constructed in such a way that the already living children obtain the same consumption bundle (that is, $\widetilde{c}_1^m(i) = \widehat{e}_1$ for each $i \in [0, \widehat{n}_1]$) while the newborn obtains $\widetilde{e}_1$ (that is, $\widetilde{c}_1^m(i) = \widetilde{e}_1$ for each $i \in [n_1(\widehat{e}_1), n_1]$). The dynasty head will choose $\widetilde{c}_0^m, \widetilde{n}_1, \widetilde{k}_1$ and $\widetilde{e}_1$ to maximize

$$\mathcal{U}_0(\mathsf{a}) = U\left(c_0^m, n_1, \left(\frac{\widehat{n}_1}{n_1}\right) U^D(\widehat{e}_1) + \left(1 - \frac{\widehat{n}_1}{n_1}\right) U^D(e_1)\right)$$

subject to the constraints

$$c_0^m + n_1\,(b + k_1) \leq \overline{e}_0; \quad \left(\frac{\widehat{n}_1}{n_1}\right)\widehat{e}_1 + \left(1 - \frac{\widehat{n}_1}{n_1}\right) e_1 \leq f(k_1); \text{and } \overline{n}_1 \geq n_1 \geq \widehat{n}_1.$$

In our example, the solution to this problem satisfies $\widetilde{n}_1 = n_1^* - \frac{\widehat{n}_1[\widehat{e}_1 - e_1^*]}{R(b - \frac{w}{R} + \frac{e_1^*}{R})} > \widehat{n}_1$ and $\widetilde{e}_1 = e_1^* < \widehat{e}_1$. Thus, even though the dynasty head would rather not discriminate among her children, she finds it optimal to do so and increase the number of children if she is constrained to provide each of her already living children with an amount of the consumption

good $\widehat{e}_1 > e_1^*$. The possibility of discriminating the children who are not born in $\widehat{a}$ makes these children cheaper, as their marginal costs –given by $b+k(\widehat{e}_1)$ in the original allocation— jump down to $b + k\left(\left(\frac{\widehat{n}_1}{\widetilde{n}_1}\right)\widehat{e}_1 + \left(1 - \frac{\widehat{n}_1}{\widetilde{n}_1}\right)e_1^*\right)$. Therefore, every Millian efficient allocation $\widehat{a}$ is $\mathcal{A}-$dominated by an asymmetric allocation $\widetilde{\mathsf{a}}$ with more individuals, in which all the people born in $\widehat{a}$ obtain at least the same utility as that obtained in $\widetilde{\mathsf{a}}$, while the dynasty head obtains strictly higher utility. Since the dynastic optimum is trivially $\mathcal{A}-$efficient, this implies, in turn, that the only symmetric, $\mathcal{A}-$efficient allocation is the dynastic optimum.

Other $\mathcal{A}-$efficient allocations. Observe that, in addition to the dynastic optimum, there are many other $\mathcal{A}-$efficient allocations. The asymmetric allocation $\widetilde{\mathsf{a}}$ constructed above (where $\widetilde{e}_1$ is chosen to maximize the dynasty head's utility) is an example of one. With the allocation $\widetilde{\mathsf{a}}$, the dynasty head cannot obtain higher utility with an allocation with more or with fewer descendants, provided she has to provide the first $\widehat{n}_1$ surviving descendants with $\widehat{e}_1$ units of the consumption good.

In Theorem 1, we show, in the context of a more general model, that in every $\mathcal{A}-$efficient allocation, a positive measure of descendants –to be more precise, the youngest individuals in every family– must devote most of their income to maximize the utility of the dynasty head. In the context of the two-period example analyzed here, Theorem 1 implies that, in every $\mathcal{A}-$efficient allocation a, the consumption scheme $\mathsf{c}_1^m(\cdot)$ must satisfy.

$$\lim_{\substack{i\to n_1 \\ i<n_1}} \mathsf{c}_1^m(i) = e_1^*. \tag{6}$$

Note that, if (6) were not satisfied, the dynasty head might obtain higher utility by altering her fertility choice to provide the youngest descendants with $\mathsf{c}_1^m(i) = e_1^*$ units of the consumption good.

This might be problematic if, in our example, the dynasty head is not altruistic and $\beta = 0$ is satisfied. In this case, the dynastic optimum, that is, the only symmetric, $\mathcal{A}-$efficient allocation, must satisfy $e_1^* = 0$. Moreover, for any other $\mathcal{A}-$efficient allocation $\widetilde{\mathsf{a}}$, a positive measure of agents obtain an arbitrarily small amount of resources. Of course, one might argue that non-altruistic preferences are rare, but, as we show in Corollary 1, the same result arises in a more general setting with infinite periods of time when the altruism of the agents extends to a *finite* number of generations of their descendants. Moreover, this type of preferences might be rare in the context of homogeneous economies in which all agents of the same generation have the same preferences, but they might be not so rare in the context of economies with heterogeneous dynasties. Suppose, for example, that we modify the example studied here and allow for several dynasty heads, with some of them exhibiting non-altruistic preferences. In an $\mathcal{A}-$efficient allocation, the consumption scheme of the descendants of such non-altruistic agents would need to satisfy $\lim_{i\to n_1} c_1^m(i) = 0$.

$\mathcal{A}-$efficiency and individual property rights. Even if the dynasty head is altruistic towards her descendants, $\mathcal{A}-$efficiency may be incompatible with the existence of property rights on the resources to be allocated in a given economy, as Schoonbroodt and Tertilt (2014) have observed. To see why, suppose we slightly modify the model described above in such a way that the amount of the consumption good k_1^o accumulated as capital by the dynasty head may be negative, in which case $d_1^0 = -k_1^o R$ represents the dynasty

head's debt in period 1. Suppose also that, in such a model, the dynastic optimum satisfies $e_1^* < w$. Finally, suppose that all descendants of the dynasty head born in any allocation are the owners of the labour capacity with which they are endowed when they are born. As none of these agents cares about the welfare enjoyed by the dynasty head, they will all use their entire labour capacity to obtain w units of the consumption good. As $e_1^* < w$ holds, the dynasty head would rather not provide any of her descendants with more resources than the resources they are endowed with. Therefore, the existence of property rights on labour time drives the economy to the Millian efficient allocation $\widehat{a}$ for which $\widehat{e}_1 = w > e_1^*$. As explained above, such an allocation is $\mathcal{A}-$inefficient. Observe also that such an allocation can only be $\mathcal{A}-$dominated by an asymmetric allocation in which *some* of the agents, the youngest children of the dynasty head, are forced to repay the debt accumulated by their parent to finance her fertility choices.

$\mathcal{A}-$efficiency in a seemingly isomorphic economy. The fact that most $\mathcal{A}-$efficient allocations are asymmetric seems to be at odds with the assumption according to which all living descendants have identical preferences and capacities. We should note, however, that the properties of $\mathcal{A}-$efficient allocations change if *i*) potential agents are distinguished from each other by the dates on which they may be born; and *ii*) each potential agent's position in the siblings' birth order is endogenous.

To clarify this, suppose that $\overline{n}$, the maximum number of children that the dynasty head can rear, represents also the length of the child-rearing period at which the dynasty head may give birth to her children. Suppose also that the dynasty head is able to decide not only the number of descendants n_1 that she bears, but also the (set of) points in time at which she wants to give birth to these descendants, represented by a Borel set $\mathsf{D}_1 \subseteq [0, \overline{n}]$ such that

$$n_1 = \int_{\mathsf{D}_1} di. \tag{7}$$

In this case, consumption *per capita* corresponding to a consumption scheme $\mathsf{c}_1^m(\cdot)$ and a fertility choice (n_1, D_1) will be given by

$$e_1 = \frac{1}{n_1} \int_{\mathsf{D}_1} \mathsf{c}_1^m(i) di. \tag{8}$$

Suppose also that the dynasty head is indifferent between any two fertility choices with the same number of children, so that her preferences on an allocation a –now described by a vector $a = (n_1, c_0^m, e_1) \in [0, \overline{n}] \times \mathbb{R}_+^2$, a consumption plan $\mathsf{c}_1^m(\cdot)$ and a fertility schedule D_1 satisfying (7) and (8)– are represented by a utility function of the form

$$\mathcal{U}_0(\mathsf{a}) = U\left(c_0^m, n_1, \frac{1}{n_1} \int_{\mathsf{D}_1} U^D(\mathsf{c}_1^m(i)) di\right).$$

If the dynasty head can choose any measurable set $\mathsf{D}_1 \subseteq [0, \overline{n}]$, dynastic optima are characterized by the same number of children n_1^* and the same consumption-fertility plan (c_0^{m*}, e_1^*, n_1^*) as the dynastic optimum arising when the dynasty head is constrained to select among sets of the form $\mathsf{D}_1 = [0, n_1]$. However, the set of $\mathcal{A}-$efficient allocations arising if the birth order is endogenous. To be more precise, it is straightforward to characterize every

$\mathcal{A}-$efficient allocation arising in this setting as a dynastic optimum, provided the length of the child-bearing period ($\overline{n}$) is twice as large as the number of children (n_1^*) corresponding to a dynastic optimum. The intuition behind this result is simple: if some of the agents born in a given allocation do not use their endowment to maximize the utility of the dynasty head, she can always replace these agents by a set of agents of equal measure that do behave as she wishes. Clearly, the allocation resulting from such a replacement constitutes an $\mathcal{A}-$improvement, since the dynasty head –the only agent living in the two allocations under comparison– will be better off with such replacement.

In GTJ's model, as in the general model described in Section 3, births also occur at different points in time, which is what allows the authors to identify potential agents by their position in their siblings' order. As in the model that describes fertility schedules as measurable sets, time is irrelevant from the point of view of the agents' preferences, but it is there. If this time is long enough to allow parents to choose the dates on which they want their children to be born, the agent's position in the siblings' order becomes endogenous, and those occupying high positions in a birth order in a given allocation might not necessarily be alive in an alternative allocation with a higher population. This, as we shall see in Theorem 2, significantly reduces the set of $\mathcal{A}-$efficient allocations.

Discrete choices. Comparison with GJT. In the example discussed above, the difference between the fertility choices corresponding to the allocations $\widetilde{\mathsf{a}}$ and $\widehat{a}$, that is always strictly positive, might become arbitrarily small as $\widehat{e}_1$ approaches e_1^*. That is, ruling out some Millian efficient allocations as being $\mathcal{A}-$inefficient may require that the population is increased in arbitrary small amounts. Therefore, the arguments used to characterize the only symmetric, $\mathcal{A}-$efficient allocation as a dynastic optimum cannot be applied when the set of fertility choices is discrete, as GJT assume in their examples.

To make this clear, suppose $n_1 \in \{0, 1, 2, 3, ..., \overline{n}\}$. In such a setting, the dynastic optimum a^* and every Millian efficient allocation $\widehat{a}$ can be characterized, respectively, as the solutions to optimization problem closely analogous to those in the definitions of $\mathcal{V}_0(\overline{e}_0)$ and $W_0(\overline{e}_0, \widehat{e}_1, U^D(\widehat{e}_1))$ –see (3) and (5), respectively–, that differ from those in that the optimal choices n_1^* and $\widehat{n}_1$ must now satisfy a constraint of the form $n_1 \in \{0, 1, 2, 3, ..., \overline{n}\}$. Thus, as in the model in which fertility choices are drawn from a *continuum*, the income e_1^* (respectively, the fertility rate n_1^*) corresponding to a^* provides a lower bound (respectively, an upper bound) for the average consumption (respectively, the fertility rate) arising with a Millian efficient allocation. Taking this into account, it is straightforward to notice that, in the discrete case, many Millian efficient allocations may be $\mathcal{A}-$efficient even though they are not dynastic optima. In particular, any Millian efficient allocation $\widehat{a}$ for which $\widehat{n}_1 = n_1^*$ must be also $\mathcal{A}-$efficient. A different question is whether or not there exist other Millian efficient allocations that are also $\mathcal{A}-$efficient and deliver a fertility choice lower than the fertility choice n_1^* corresponding to the dynastic optimum. GJT (2007, Ex.2) provide an example in which this occurs.

We should note, however, that this possibility arises only if potential agents are distinguished from one another by the their positions in the siblings' order and not by their birth dates. In the latter case, $\mathcal{A}-$efficiency can be characterized as dynastic optimality even when the set of fertility choices –that is, the set of dates at which they dynasty head's descendants may be born– is discrete.

$\mathcal{P}-$efficient allocations. The notion of $\mathcal{P}-$efficiency is associated to a utility function $\mathcal{U}_1^N$, determining the utility $\mathcal{U}_1^N(\mathsf{a}, i)$ attributed to the i'th descendant in case that this descendant is unborn. Complemented with the utility function defined by $\mathcal{U}_1(\mathsf{a}, i) = \mathsf{c}_1^m(\mathsf{i})$, that represents the preferences of every descendant i in those allocations in which i is alive, the function $\mathcal{U}_1^N$ provides a complete description of the agents' preferences across allocations. An allocation $\widehat{\mathsf{a}}$ is $\mathcal{P}-$efficient if there does not exist an alternative allocation a unanimously preferred by all potential agents in the economy and strictly preferred by a positive measure of potential agents.

In all their applications, GJT study the properties of $\mathcal{P}-$efficient allocations when the function $\mathcal{U}_1^N$ is constant and $\mathcal{U}_1^N(\mathsf{a}, i) = \overline{e}_1$. If, for example, $\overline{e}_1 < 0$ is satisfied, then the only Millian efficient allocation that is also $\mathcal{P}-$efficient is the dynastic optimum, that is also the only allocation arising as $\mathcal{A}-$efficient if we identify potential agents with the *Birth-Date criterion*. There are, however, many other allocations that are $\mathcal{P}-$efficient. For any fertility level $\breve{n}_1$ above the fertility level n_1^* corresponding to the dynastic optimum, the allocation that maximizes the utility of the dynasty head among all allocations for which $n_1 \geq \breve{n}_1$ must be $\mathcal{P}-$efficient. Therefore, fertility levels arising with $\mathcal{P}-$efficient allocations can be arbitrarily high. As the utility attributed to the unborn $\overline{e}$ is higher, all Millian efficient allocations for which $e_1^* \leq \widehat{e}_1 \leq \overline{e}_1$ are also $\mathcal{P}-$efficient, and other allocations that are not Millian efficient might be $\mathcal{P}-$efficient as well.

The notion of $\mathcal{P}-$efficiency –with constant utility levels attributed to the unborn– brings with it another difficulty: Who sets $\overline{e}$? To avoid this difficulty, GTJ (2007, p.1054) suggest to take $\overline{e}_1 = e_1^*$, which leaves the dynastic optimum as the only $\mathcal{P}-$efficient allocation. In Section 4, we shall explore the possibility that the utility attributed to the unborn depends on the utility obtained by their living siblings.

3 Feasible fertility choices in a general, OLG setting

3.1 A benchmark framework

Throughout the remaining of the paper, we shall focus on a particular class of overlapping generations economies with infinite periods of time. Each individual in an economy lives for at most three of these periods, so that individuals living at $t = 0$, 1, 2,... are referred to as *children, middle-aged adults* or *old adults* depending on whether it is their first, second or third period of life. As in GJT, the set of potential agents that are actually alive at any given period is endogenous and depends on the fertility plans selected by the agents. In contrast with these authors, we assume that each middle-aged adult *potentially* alive at $t = 1$ is identified by a non-negative number $i_1 \in [0, \overline{n}]$ determining the instant, in the child-rearing period $[0, \overline{n}]$, in which the agent may be born. For $t = 1$, 2, 3,..., each middle-aged adult potentially alive at t is identified by a vector $i^t = (i^{t-1}, i_t) \in [0, \overline{n}]^t$, where i_t specifies the instant at which the agent may be born and $i^{t-1} = (i_1, ..., i_{t-1})$ identifies the agent's parent. To simplify things, all agents belong to the same dynasty, initiated by the only agent who is middle aged at $t = 0$, the *dynasty head*, hereafter represented by i^0. Let $\mathcal{B}[0, \overline{n}]^t$ the class of Borel sets in $[0, \overline{n}]^t$. For every set $D \in \mathcal{B}[0, \overline{n}]$, the Lebesgue measure of D will be denoted by $\mu\{D\} \equiv \int_D di$; while, for every set $D^t \in \mathcal{B}[0, \overline{n}]^t$ of potential middle-aged agents at t, the Lebesgue measure of D^t will be denoted by $\mu^t\{D^t\} \equiv \int_D di^t$. For intervals

of the form $D^t = [a^t, b^t] \subset \mathcal{B}[0,\overline{n}]^t$, the Lebesgue integral $\int_{[a^t,b^t]} di^t$ may also be written as $\int_{[a^t,b^t]} di^t = \int_{a^t}^{b^t} di^t$.

Differently from GJT, the set of feasible fertility choices is represented by a set $\mathcal{D} \subseteq \mathcal{B}[0,\overline{n}]$. Thus, although potential agents will be identified, in general, with the *Birth-Date criterion*, we allow, as a particular specification, that the set of fertility choices adopts the form $\mathcal{D}^O \equiv \{[0,n] : n \in [0,\overline{n}]\}$. Note that, in this case, the agents' birth dates and their positions in their siblings' birth order coincide. In what follows, we shall assume that $\mathcal{D}^O \subseteq \mathcal{D}$ is satisfied.

A fertility plan D is a sequence of mappings $\mathsf{D} = \left\{\mathsf{D}_{t+1} : [0,\overline{n}]^t \longrightarrow \mathcal{D}\right\}_{t\geq 0}$. Each mapping $\mathsf{D}_{t+1} : [0,\overline{n}]^t \longrightarrow \mathcal{D}$ represents a schedule determining, for each $i^t \in [0,\overline{n}]^t$, the set of sub-periods $\mathsf{D}_{t+1}(i^t)$ in the child-rearing period in which agent i^t's children are born. We assume that, at each point in time, only one child may be born, so that the number of children that agent i^t decides to bear, which we denote by $\mathsf{n}_{t+1}(i^t)$, is given by

$$\mathsf{n}_{t+1}(i^t) \equiv \mu\left\{\mathsf{D}_{t+1}(i^t)\right\} = \int_{\mathsf{D}_{t+1}(i^t)} di. \tag{9}$$

For each t and every $i^t = (i^{t-1}, i_t) \in [0,\overline{n}]^t$, agent i^t is said to be alive with fertility plan D if agent i^{t-1} is also alive and $i_t \in \mathsf{D}_t(i^{t-1})$ is satisfied. For every individual $i^t \in \mathbb{R}_+^t$ and every $\tau \geq t+1$, the set of descendants of i^t at their middle age at τ is denoted by $\mathsf{D}^\tau(i^t)$. The set of middle-aged adults actually living at t with a fertility plan D is denoted by $\mathsf{D}^t(i^0)$ and its measure is given by

$$\mu^t\left\{\mathsf{D}^t(i^0)\right\} = \int_{\mathsf{D}^t(i^0)} di^t = \int_{\mathsf{D}^{t-1}(i^0)} \left(\int_{\mathsf{D}_t(i^{t-1})} di\right) di^{t-1} = \int_{\mathsf{D}^{t-1}(i^0)} \mathsf{n}_t(i^{t-1}) di^{t-1}.$$

In addition to children, there is only one homogenous good produced at every period $t \geq 1$. This consumption good is produced at each period $t \geq 0$ using, as inputs, a given amount K_t of the same good invested in the previous period $t-1$ as physical capital and a given amount of labour L_t provided by middle-aged adults. That is, $Y_t \leq F_t(K_t, L_t)$, where Y_t is total output and $F_t : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ exhibits constant-returns to scale and it is non-decreasing, concave and continuously differentiable. Rearing children is a production activity that takes place within each household and its costs are represented by a strictly increasing, convex and continuously differentiable function $b_t : [0,\overline{n}] \longrightarrow \mathbb{R}_+$. Thus, a middle-aged adult who decides to rear n_{t+1} children at period t needs to spend $b_t(n_{t+1})$ units of the consumption good. Fertility and consumption plans of potential agents are represented by a fertility plan D and a sequence of integrable functions $\mathsf{c} = \left\{(\mathsf{c}_t^m, \mathsf{c}_{t+1}^o) : [0,\overline{n}]^t \longrightarrow \mathbb{R}_+^2\right\}_{t\geq 0}$ that determines, for each $t \geq 0$ and each potential agent $i^t \in \mathbb{R}_+^t$, the consumption vector $(\mathsf{c}_t^m(i^t), \mathsf{c}_{t+1}^o(i^t))$ chosen by agent i^t through her life cycle. Thus, it is assumed that children do not take consumption decisions.

The resource constraint faced by potential agents is described as follows. At time $t = 0$, the amount of resources available to finance consumption $(\mathsf{c}_0^m(i^0))$, fertility $(\mathsf{n}_1(i^0))$ and investment decisions $(\mathsf{k}_1^o(i^0))$ of the dynasty head is bounded by an initial endowment $\overline{e}_0$ available for the dynasty head, that is,

$$\mathsf{c}_0^m(i^0) + b_0(\mathsf{n}_1(i^0)) + \mathsf{k}_1^o(i^0) \leq \overline{e}_0. \tag{10}$$

For each period $t \geq 0$, each living agent is endowed with 1 unit of labour time when he/she reaches middle age. Then, labour is supplied inelastically, so that labour supply at any given period coincides with the measure of middle-aged agents alive at t, that is, $L_t = \mu^t \{\mathsf{D}^t(i^0)\}$. By writing, for each t and each $i^t \in \mathsf{D}^t(i^0)$, $\mathsf{k}^o_{t+1}(i^t)$ for $\mathsf{k}^o_{t+1}(i^t) = \mathsf{n}_{t+1}(i^t)\frac{K_{t+1}}{\mu^t\{\mathsf{D}^t(i^0)\}}$, the resource constraint at each date $t \geq 1$ is

$$\int_{\mathsf{D}^{t-1}(i^0)} \mathsf{c}^o_t(i^{t-1})di^{t-1} + \int_{\mathsf{D}^t(i^0)} \left[\mathsf{c}^m_t(i^t) + b_t(\mathsf{n}_{t+1}(i^t)) + \mathsf{k}^o_{t+1}(i^t)\right] di^t \leq \int_{\mathsf{D}^{t-1}(i^0)} F_t(\mathsf{k}^o_t(i^{t-1}), \mathsf{n}_t(i^{t-1}))di^{t-1}. \tag{11}$$

In what follows, an allocation $\mathsf{a} = (\mathsf{D}, \mathsf{c}, \mathsf{k}^o)$ is a fertility plan D, a consumption plan $\mathsf{c} = \left\{(\mathsf{c}^m_t, \mathsf{c}^o_{t+1}) : [0, \overline{n}]^t \longrightarrow \mathbb{R}^2_+\right\}_{t\geq 0}$ determining consumption choices of every potential agent and an investment plan $\mathsf{k}^o = \left\{\mathsf{k}^o_{t+1} : [0, \overline{n}]^t \longrightarrow \mathbb{R}_+\right\}_{t\geq 0}$ determining investment decisions in every period. An allocation a is feasible if it satisfies the initial condition in (10), the resource constraint in (11), as well as condition (9). The set formed by all feasible allocations is denoted by $\mathcal{F}$.

Write $\mathbb{R}^*$ for the set of extended real numbers $\mathbb{R}^* \equiv \{-\infty\} \cup \mathbb{R}$, and write $\mathsf{x}_t(i^t)$ for the consumption-fertility bundle

$$\mathsf{x}_t(i^t) = (\mathsf{c}^m_t(i^t), \mathsf{c}^o_{t+1}(i^t), n_{t+1}(i^t)).$$

Throughout the paper, we assume that preferences of every potential agent of generation t on the set of allocations in which the agent is alive are represented by a utility function $\mathcal{U}_t : \mathcal{F} \times \mathbb{R}^* \longrightarrow \mathbb{R}^*$ satisfying, for every $\mathsf{a} \in \mathcal{F}$ and $i^t \in \mathsf{D}^t(i^0)$,

$$\mathcal{U}_t(\mathsf{a}; i^t) = U\left(\mathsf{x}_t(i^t), \frac{1}{\mathsf{n}_{t+1}(i^t)} \int_{\mathsf{D}_{t+1}(i^t)} \mathcal{U}^D_{t+1}(\mathsf{a}; i^t, i_{t+1})di_{t+1}\right), \tag{12}$$

where $U : \mathbb{R}^3_+ \times \mathbb{R}^* \longrightarrow \mathbb{R}^*$ is non-decreasing, concave and continuously differentiable and $\mathcal{U}^D_{t+1}$ is a function representing the agents' preferences on fertility and consumption choices made by his/her living descendants. Thus, as the functions $\mathcal{U}_t$ and $\mathcal{U}^D_t$ may not coincide, parents' preferences regarding their children's decisions might differ from the preferences of the children themselves.

We shall assume that, for each t, $\mathcal{U}^D_t : \mathcal{F} \times \mathbb{R}^t_+ \longrightarrow \mathbb{R}^*$ is recursively defined, for every $\mathsf{a} \in \mathcal{F}$ and every $i^t \in \mathsf{D}^t(i^0)$, by

$$\mathcal{U}^D_t(\mathsf{a}; i^t) = U^D\left(\mathsf{x}_t(i^t), \frac{1}{\mathsf{n}_{t+1}(i^t)} \int_{\mathsf{D}_{t+1}(i^t)} \mathcal{U}^D_{t+1}(\mathsf{a}; i^t, i_{t+1})di_{t+1}\right), \tag{13}$$

where $U^D : \mathbb{R}^3_+ \times \mathbb{R}^* \longrightarrow \mathbb{R}^*$ is also non-decreasing, concave and continuously differentiable. The fact that each function $\mathcal{U}^D_t$ is defined recursively implies that the preferences of any agent of generation t and those of their children regarding the consumption and fertility choices of any common descendant coincide.

3.2 On types of altruism.

Our general setting admits a wide range of particular specifications, or *environments*, frequently studied in the literature of endogenous fertility.

Dynastic altruism. By an environment with dynastic –or *perfect*– altruism, we shall refer to the class of particular specifications of the model for which $U^D \equiv U$ and U is strictly increasing in c_t^m, n_{t+1} and u_{t+1}^D. In this particular environment, every agent cares about the utility of their immediate descendants, which, proceeding recursively, implies that every agent cares about consumption and fertility decisions of all her descendants. Observe that we are not imposing that $U(\cdot)$ is strictly monotonic in c_{t+1}^o, which allows us to consider, as different specifications of dynastic altruism, *i*) models in which the agents live for one period and provide with bequests to their immediate descendants, as well as *ii*) models in which the different generations of agents are truly overlapping and parents provide their immediate descendants with gifts. Examples of the first type of models –although they both restrict their analysis to symmetric allocations— are the pioneering work by Razin and Ben-Zion (1975), for whom

$$U\left(x_t, u_{t+1}^D\right) = U^D(x_t, u_{t+1}^D) = v(c_t^m) + \gamma(n_{t+1}) + \beta u_{t+1}^D;$$

as well as the model developed in Barro and Becker (1989), for whom

$$U\left(x_t, u_{t+1}^D\right) = U^D(x_t, u_{t+1}^D) = v(c_t^m) + \alpha(n_{t+1}) n_{t+1} u_{t+1}^D;$$

with $\alpha(n_{t+1}) = \alpha n_{t+1}^{-\epsilon}$ being the endogenous discount rate. Finally, an example of a model with dynastic altruism and truly overlapping generations is Schoonbroodt and Tertilt (2014), for whom

$$U\left(x_t, u_{t+1}^D\right) = U^D(x_t, u_{t+1}^D) = v(c_t^m) + \beta v(c_{t+1}^o) + \Psi(n_{t+1}, u_{t+1}^D).$$

No altruism. In many other models studying fertility, the agents are not altruistic at all, and children are viewed as a consumption good.[4] Since we are not imposing that U or U^D must be strictly monotonic in u_{t+1}^D, a setting with no altruism is a particular specification of our general framework, for which $U\left(x_t, u_{t+1}^D\right) = u(x_t)$ and $U^D\left(x_t, u_{t+1}^D\right) = u(x_t)$.[5]

Non-dynastic altruism. But there exist other possibilities. In the exogenous fertility literature, some authors[6] have studied environments with *limited*, or *non-dynastic* altruism to study to what extent the positive (for example, *Ricardian Equivalence*) or normative (efficiency) properties of the equilibria arising with dynastic altruism can be extended to more general settings. Endogenous fertility literature also has abundant specifications of altruism in which the *quality* of children, from which parents derive utility, is not necessarily identified with the children's utilities, and may take the form of goods spent on each child, as in

[4]Examples of this approach –focusing exclusively on symmetric allocations– are Eckstein and Wolpin (1985), Michel and Wigniolle (2007) or Conde-Ruiz *et al.* (2010).

[5]Although, with the description of preferences given in (12), the specification of the function U^D is irrelevant if U satisfies $U(x_t, u_{t+1}^D) = u(x_t)$, by imposing that $U(x_t, u_{t+1}^D) = U^D(x_t, u_{t+1}^D) = u(x_t)$ is satisfied in settings with no altruism we may assume, without loss of generality, that each function $\mathcal{U}_t^D$ as defined in (13) represents the preferences of the agents born at t in this type of environments, which simplifies the statement of Theorem 1.

[6]See, e.g., Bernheim and Ray (1989) and the references therein.

Becker and Lewis (1973); income, as in Galor and Weil (2000); human capital, as in De la Croix and Doepke (2005); or consumption, as in Kollmann (1997).[7]

A particular specification of non-dynastic altruism is that for which an agent's altruism extends towards all her future descendants, which corresponds, for example, to the case in which $U^D(x_t, u^D_{t+1}) = U(x_t, \beta u^D_{t+1})$, with $0 < \beta < 1$ and U –and, hence, U^D–being strictly increasing in u^D_{t+1}. We shall refer to this type of altruism as *infinite-horizon, non-dynastic altruism*. However, the function U^D needs not be strictly increasing in u^D_{t+1}, that is, the agents might be altruistic only towards their immediate descendants. We shall refer to this type of non-dynastic altruism as *finite-horizon, non-dynastic altruism*, which is represented by utility functions of the form

$$U\left(x_t, u^D_{t+1}\right) = v(x_t) + \delta u^D_{t+1} \text{ and } U^D(x_t, u^D_{t+1}) = v(x_t);$$

with v being strictly increasing in c^m_t and n_{t+1} and $\delta \in (0, 1)$.

Although the preferences considered in the present paper allow for non-dynastic altruism, we shall impose two additional assumptions on preferences, ensuring that the agents' preferences and those of their parents are, in a sense that we clarify below, consistent.[8] The first of these assumptions imposes that, keeping fixed the total amount of resources available to any given agent and the decisions taken by the agent's descendants, the agent's preferences on how to distribute these resources among consumption, fertility and investment coincide with those of her parents. Formally:

A1 For any fixed $u^D \in \mathbb{R}^*$ and any two $(x, \widetilde{x}) \in \mathbb{R}^3_+ \times \mathbb{R}^3_+$, $U^D\left(x, u^D\right) \geq U^D\left(\widetilde{x}, u^D\right)$ is satisfied whenever $U\left(x, u^D\right) \geq U\left(\widetilde{x}, u^D\right)$ is satisfied.

The second assumption imposes that the agents discount the utility obtained by their grandchildren at least at the same rate as their parents do, which ensures that, whenever an agent is willing to increase the total resources available to any of her grandchildren – and, hence, to increase the utility that the agent obtains from consumption decisions of her grandchildren–, then the agent's children agree on that decision. Formally:

A2 For any two $\left(x, u^D\right) \in \mathbb{R}^3_+ \times \mathbb{R}^*$ and $\left(\widetilde{x}, \widetilde{u}^D\right) \in \mathbb{R}^3_+ \times \mathbb{R}^*, U\left(x, u^D\right) > U\left(\widetilde{x}, \widetilde{u}^D\right)$ is satisfied whenever $U^D\left(x, u^D\right) > U^D\left(\widetilde{x}, \widetilde{u}^D\right)$ is satisfied.

Observe that all the examples of the different specifications of preferences given above satisfy Assumptions *A*1 and *A*2.

[7]Since we are imposing that the agents' utilities depends exclusively on their consumption decisions –as in Kollmann's paper– we rule out other specifications of non-dynastic altruism in which the agents are concerned on their descendants investment decisions or on their wealth. However, as we discuss in the Supplementary Appendix S.A, our characterizations of Millian efficient allocations or dynastic optima can be easily extended to environments with non-dynastic altruism in which utility depends on –human– capital investments or wealth.

[8]These conditions are needed to obtain the necessary conditions $\mathcal{A}-$efficiency in Theorem 1 and are also needed to establish Theorem 3, that characterizes symmetric, $\mathcal{P}-$efficient allocations as Millian efficient allocations.

3.3 Dynastic optima and value functions

Since we are assuming that $\mathcal{D}^O \subseteq \mathcal{D}$ is satisfied and the agents care only about the number of children they bear and not about the specific points in time at which their children are born, the maximum utility that the dynasty head can obtain with a feasible allocation is not affected by the constraints in fertility choices that parents may face. To characterize dynastic optima, for every allocation $\mathsf{a} \in \mathcal{F}$ and every $t \geq 0$ and $i^t \in \mathsf{D}_t(i^0)$, write $\mathsf{e}_t(i^t)$ for the amount of physical resources –or *income*– available to finance agent i^t's consumption, fertility and investment decisions at period t; that is,

$$\mathsf{e}_t(i^t) := \mathsf{c}^m_t(i^t) + b_t(\mathsf{n}_{t+1}(i^t)) + \mathsf{k}^o_{t+1}(i^t).$$

Consider now an arbitrary i^t and an arbitrary e_t, and let $\mathcal{F}(e_t; i^t)$ be the set formed by all sequences

$$\mathsf{a}_t = \{(\mathsf{c}^m_\tau, \mathsf{c}^o_{\tau+1}, \mathsf{k}^o_{\tau+1}, \mathsf{D}_{\tau+1}) : [0, \overline{n}]^\tau \longrightarrow \mathbb{R}^3_+ \times \mathcal{D}\}_{\tau \geqslant t}$$

satisfying $\mathsf{e}_t(i^t) \leq e_t$ as well as the feasibility constraints that all potential descendants of agent i^t would face at τ if they were not allowed to obtain resources from other agents in the economy, that is, satisfying

$$\int_{\mathsf{D}^{\tau-1}(i^t)} \mathsf{c}^o_\tau(i^{\tau-1}) di^{\tau-1} + \int_{\mathsf{D}^\tau(i^t)} \left[\mathsf{c}^m_\tau(i^\tau) + b_\tau(\mathsf{n}_{\tau+1}(i^\tau)) + \mathsf{k}^o_{\tau+1}(i^\tau)\right] di^\tau \leq \int_{\mathsf{D}^{\tau-1}(i^t)} F_\tau(\mathsf{k}^o_\tau(i^{\tau-1}), \mathsf{n}_\tau(i^{\tau-1})) di^{\tau-1},$$

for $\tau \geq t$. Also, for each $t \geq 1$ and $e_t \geq 0$, let $\mathcal{V}^D_t(e_t)$ be defined as the maximum utility that the dynasty head can obtain from their descendants born at t by endowing any of their immediate descendants with e_t units of resources, that is[9]

$$\mathcal{V}^D_t(e_t) := \max_{i^t \in [0,\overline{n}]^t} \left\{ \max_{\mathsf{a}_t \in \mathcal{F}(e_t; i^t)} \mathcal{U}^D_t(\mathsf{a}_t; i^t) \right\}. \tag{14}$$

With this notation, the maximum utility that the dynasty head can obtain with a feasible allocation can be characterized as the solution to

$$\mathcal{V}_0(\overline{e}_0) = \max_{\substack{(k^o_1, x_0) \in \mathbb{R}^3_+ \times [0,\overline{n}] \\ \mathsf{e}_1 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+}} \Big\{ \; U\left(x_0, \frac{1}{n_1}\int_0^{n_1} \mathcal{V}^D_1(\mathsf{e}_1(i)) di\right) : \; c^m_0 + b_0(n_1) + k^o_1 \leq \overline{e}_0; \\ c^o_1 + \int_0^{n_1} \mathsf{e}_1(i) di \leq F_1(k^o_1, n_1)\Big\}. \tag{15}$$

Throughout the paper, it is assumed that $\mathcal{V}_0(\overline{e}_0)$ is well defined.

3.4 Symmetric allocations.

As preferences and labour capacities of any two agents of the same generation are identical, it seems innocuous, both from normative and positive concerns, to restrict attention to *ex-post symmetric* allocations, that is, to allocations for which any two agents

[9]Since the utility received by the dynasty head from consumption of any of her descendants is the same, any choice of i^{t+1} in the optimization problem in the definition of $\mathcal{V}^D_t(e_t)$ is optimal.

of the same generation choose the same consumption and fertility bundles. Formally, a feasible allocation $\mathsf{a} \in \mathcal{F}$ is said to be *ex-post symmetric* (or, simply, *symmetric*) if for any t and any two agents i^t, $\tilde{i}^t \in \mathsf{D}^t(i^0)$ one has $\mathsf{x}_t(i^t) = \mathsf{x}_t(\tilde{i}^t) = x_t$ and $\mathsf{k}^o_{t+1}(i^t) = \mathsf{k}^o_{t+1}(\tilde{i}^t) = k^o_{t+1}$. A symmetric allocation is, therefore, represented by a pair of sequences $a \equiv \left\{(k^o_{t+1}, x_t) \in \mathbb{R}^3_+ \times [0, \overline{n}]\right\}_{t=0,1,2...}$ satisfying the initial condition $c^m_0 + b_0(n_1) + k^o_1 \leq \overline{e}_0$; and, for each $t \geq 1$, the feasibility constraint

$$c^o_t + n_{t+1}\left[c^m_t + b_t(n_{t+1}) + k^o_{t+1}\right] \leq F_t(k^o_t, n_t). \tag{16}$$

Denote by $\mathcal{S}$ the set containing all feasible, symmetric allocations. Note that for every t, the utility obtained by the dynasty head from consumption and fertility decisions of the living agents with a symmetric allocation a satisfies $\mathcal{U}^D_t(\mathsf{a}; \tilde{i}^t) = U^D_t(a)$, where $U^D_t : \mathcal{S} \longrightarrow \mathbb{R}^*$ is recursively defined, for each t, by $U^D_t(a) = U^D\left(x_t, U^D_{t+1}(a)\right)$. The utility obtained by an agent of generation t with a symmetric allocation is $U_t(a) = U(x_t, U^D_{t+1}(a))$.

Even when we restrict the analysis to symmetric allocations, the feasibility constraint in (16) involves the product of two endogenous variables, which implies that the set of feasible allocations is non convex. A *Millian efficient* allocation is a symmetric allocation that is not $\mathcal{A}-$dominated by any other symmetric allocation.[10]

Some allocations can be regarded as being symmetric in a weaker, *ex-ante* sense. Given a sequence e of income schemes corresponding to a feasible allocation a, for every $t \geq 0$, $i^t \in \mathsf{D}^t(i^0)$, let $E^{\mathsf{e}}_{t+1} : \mathbb{R}^{t+1} \longrightarrow [0,1]$ be the function determining, for each $i^t \in \mathbb{R}^t$, the distribution of income $E^{\mathsf{e}}_{t+1}(\cdot, i^t)$ among i^t's immediate descendants; that is, E^{e}_{t+1} is defined,[11] for each $(e, i^t) \in \mathbb{R}^{t+1}$, by

$$E^{\mathsf{e}}_{t+1}(e, i^t) = \frac{\mu\left\{i \in \mathsf{D}_{t+1}(i^t) : \mathsf{e}_{t+1}(i^t, i) \leq e\right\}}{\mu\left\{\mathsf{D}_{t+1}(i^t)\right\}}.$$

Thus, $E^{\mathsf{e}}_{t+1}(e, i^t)$ determines the probability that a randomly-chosen, immediate descendant of i^t spends –in consumption, fertility and investment decisions– at most e units of the homogeneous good at time $t+1$ with the income scheme $\mathsf{e}_{t+1}(i^t, \cdot)$.

With this notation, an allocation a will be referred to as *ex-ante* symmetric if the distribution of income among any agent's descendants is determined randomly, so that the income accumulated by, say, the eldest n children of an agent is the same as the average by the youngest $[\mathsf{n}_{t+1}(i^t) - n]$ children. That is, if for each $t = 0, 1, 2...$each $i^t \in \mathbb{R}^t$ and each $D_{t+1} \subseteq \mathsf{D}_{t+1}(i^t)$ one has

$$\int_{D_{t+1}} \mathsf{e}_{t+1}(i^{t+1}) di^{t+1} = \mu\left\{D_{t+1}\right\} \int_{\mathbb{R}_+} e dE^{\mathsf{e}}_{t+1}(e, i^t).$$

[10] We include a characterization of Millian efficient allocations –that takes into account the non-convexities mentioned above– in the Supplementary Appendix S.A.

[11] Observe that, as we are assuming that, in a feasible allocation, $\mathsf{e}_{t+1}(i^{t+1}) = 0$ whenever $i^{t+1} \notin \mathsf{D}_{t+1}(i^t)$, the measure of the set $\mathsf{D}_{t+1}(i^t)$ can be written equivalently as an integral depending only on e, that is

$$\mu\left\{\mathsf{D}_{t+1}(i^t)\right\} = \int\limits_{i:\mathsf{e}_{t+1}(i^t,i)>0} di.$$

In fact, dynastic maximization may require randomization. To see this, let $\Delta\mathbb{R}_+$ be defined as the set formed by all non-decreasing, measurable (distribution) functions $E : \mathbb{R}_+ \longrightarrow [0,1]$ satisfying $\lim_{e\to\infty} E(e) = 1$. Taking this into account, it is straightforward to show that the expenditure function $E_1^{\mathsf{e}^*}(\cdot, i^0)$ corresponding to a dynastic optimum must solve

$$\mathcal{V}_0(\overline{e}_0) = \max_{E:\mathbb{R}_+\longrightarrow[0,1]\in\Delta\mathbb{R}_+} \left\{ W_0\left(\overline{e}_0, \int_{\mathbb{R}_+} e dE(e), \int_{\mathbb{R}_+} \mathcal{V}_{t+1}^D(e) dE(e)\right)\right\}, \tag{17}$$

where $W_0 : \mathbb{R}^2 \times \mathbb{R}^* \to \mathbb{R}_+$ is defined,[12] for each $(\overline{e}_0, e_1, u_1^D)$, by

$$W_0(\overline{e}_0, e_1, u_1^D) = \max_{(k_{\tau+1}^o, x_\tau)\in\mathbb{R}_+^3\times[0,\overline{n}]} \left\{ U(x_0, u_1^D) :\ c_0^m + b_0(n_1) + k_1^o \leq \overline{e}_0;\ c_1^o + n_1 e_1 \leq F_1(k_1^o, n_1)\right\}.$$

Also, by writing $\overline{\mathcal{V}}_{t+1}^D(e_{t+1})$ for the maximum utility that the dynasty head obtains by providing a positive measure of her immediate descendants with an average income e_{t+1}, that is,

$$\overline{\mathcal{V}}_{t+1}^D(e_{t+1}) = \max_{E:\mathbb{R}_+\longrightarrow[0,1]\in\Delta\mathbb{R}_+} \left\{ \int_{\mathbb{R}_+} \mathcal{V}_{t+1}^D(e) dE(e) : \int_{\mathbb{R}_+} e dE(e) = e_{t+1}\right\}, \tag{18}$$

the value function $\mathcal{V}_0$ can be written as

$$\mathcal{V}_0(\overline{e}_0) = \max_{e_1\geq 0} \left\{ W_0\left(\overline{e}_0, e_1, \overline{\mathcal{V}}_1^D(e_1)\right)\right\}. \tag{19}$$

With this representation, it becomes clear that, although dynastic optima may be always symmetric in the weaker, *ex-ante* sense, they might be non symmetric in the stronger, *ex-post* sense. Dynastic optima will be (*ex-post*) non-symmetric when the sequence of value functions $\left\{\mathcal{V}_{t+1}^D\right\}_{t\geq 0}$ differ from the sequence of value functions $\left\{V_{t+1}^D\right\}_{t\geq 0}$ arising from a dynastic maximization problem in which the dynasty head is restricted to select (*ex-post*) symmetric allocations.[13] To be more precise, let $\widehat{a}$ be any allocation maximizing the utility of the dynasty head among symmetric allocations and suppose that there exists a period $t \geq 0$ for which $\mathcal{V}_{t+1}^D(\widehat{e}_{t+1}) > V_{t+1}^D(\widehat{e}_{t+1})$. In this case, a dynastic optimum cannot be (*ex-post*) symmetric. It is straightforward to show, using (17) and Jensen's inequality, that a sufficient condition ensuring that dynastic optima are ex-post symmetric is that each value function $\mathcal{V}_{t+1}^D$ is concave.

Observe, on the other hand, that each value function $\overline{\mathcal{V}}_{t+1}^D$ must be concave[14] and, hence, absolutely continuous and differentiable almost everywhere.[15] Taking this into account, it

[12] In models in which the agents live and consume only for one period (that is, in which $c_{t+1}^o(i^t) = 0$), the definition of the indirect utility function W_0 coincides with the one given in the two-period example of Section 2, equation (5).

[13] The properties of the constrained value functions $\left\{V_{t+1}^D\right\}_{t\geq 0}$ are studied in the Supplementary Appendix S.A.

[14] Note that if a distribution function E_{t+1} solves the optimization problem in the definition of $\overline{\mathcal{V}}_{t+1}^D(e_{t+1})$ and a distribution function E'_{t+1} solves the optimization problem in the definition of $\overline{\mathcal{V}}_{t+1}^D(e'_{t+1})$, then, for any $\alpha \in (0,1)$, the distribution function $E_{t+1}^\alpha \equiv \alpha E_{t+1} + (1-\alpha) E'_{t+1}$ is feasible in the optimization problem in the definition of $\overline{\mathcal{V}}_{t+1}^D\left(\alpha e_{t+1} + (1-\alpha) e'_{t+1}\right)$, which establishes that implies that $\overline{\mathcal{V}}_{t+1}^D$ must be concave.

[15] See, e.g., Theorem 10 in Royden (1988, p.107).

follows from (19) that a sufficient condition ensuring the concavity of the value function $\mathcal{V}_0$ is that the indirect utility function W_0 is also concave. Unfortunately, the non-convexities appearing in the feasibility constraints in the definition of W_0 may give rise to non-concave value functions.[16]

4 $\mathcal{A}-$EFFICIENCY

In this section, we explore the properties of $\mathcal{A}-$efficient allocations in the context of the general framework described in Section 3. In this context, an allocation $\widehat{\mathsf{a}}$ is $\mathcal{A}-$efficient if there does not exist a feasible allocation a such that i) for every $t \geq 0$ and every $i^t \in \widehat{\mathsf{D}}^t(i^0) \cap \mathsf{D}^t(i^0)$ one has $\mathcal{U}_t(\mathsf{a}; i^t) \geq \mathcal{U}_t(\widehat{\mathsf{a}}; i^t)$; and, ii) there exists a period τ and a set $\mathsf{D}^\tau \subseteq \widehat{\mathsf{D}}^\tau(i^0) \cap \mathsf{D}^\tau(i^0)$ of positive measure for which $\mathcal{U}_\tau(\mathsf{a}; i^\tau) > \mathcal{U}_\tau(\widehat{\mathsf{a}}; i^\tau)$ holds for every $i^\tau \in \mathsf{D}_\tau$.

Observe that this definition of $\mathcal{A}-$efficiency seems to coincide with the one provided by GTJ (2007, p.1047) but in fact it does not. In our model, potential agents are identified, in general, with the *Birth-Date criterion* and their positions in their siblings' birth order are endogenous. Yet, our framework allows for a constrained set of fertility choices $\mathcal{D}$, so that, if $\mathcal{D} \equiv \mathcal{D}^O$ holds, then the birth date of an agent coincides with his/her position in the birth order, and the definition of $\mathcal{A}-$efficiency provided by GTJ coincides with ours. If $\mathcal{D}^O \subseteq \mathcal{D}$ is satisfied, then an allocation that is $\mathcal{A}-$efficient with the *Birth-Date criterion* must also be $\mathcal{A}-$efficient with the *Birth-Order criterion*, although, in general, the converse is not true.

In Theorem 1, we provide a necessary condition for $\mathcal{A}-$efficiency that applies to all specifications of the set of feasible fertility choices that satisfy $\mathcal{D}^O \subseteq \mathcal{D}$ and, therefore, to the two criteria to identify potential agents considered in this paper. Given an allocation $\widehat{\mathsf{a}}$, for each t and each $i^t \in \mathsf{D}^t(i^0)$, write $\hat{\imath}^y_{t+1}(i^t)$ for the birth date of the youngest descendant of i^t in the allocation $\widehat{\mathsf{a}}$, that is,

$$\hat{\imath}^y_{t+1}(i^t) = \sup\left\{i : i \in \widehat{\mathsf{D}}_{t+1}(i^t)\right\}.$$

Assume, without loss of generality, that in any feasible allocation a, $\mathsf{e}_t(i^t) = 0$ and $\mathcal{U}^D_t(\mathsf{a}; i^t) = 0$ hold for every $i^t \notin \mathsf{D}^t(i^0)$. Taking this into account, the integral $\displaystyle\int_{\mathsf{D}_{t+1}(i^t)} \mathcal{U}^D_{t+1}(\mathsf{a}; i^t, i)di$ can be equivalently written as

$$\int_{\mathsf{D}_{t+1}(i^t)} \mathcal{U}^D_{t+1}(\mathsf{a}; i^t, i)di^i = \int_0^{\hat{\imath}^y_{t+1}(i^t)} \mathcal{U}^D_{t+1}(\mathsf{a}; i^t, i)di.$$

With this notation, Theorem 1 states that, in every $\mathcal{A}-$efficient allocation, some of the agents –specifically, the youngest within each family– must devote most of their entire income to maximize not their own utility, but their parents' utility and, hence, the dynasty head's utility. All the proofs in the paper are relegated to the Appendix.

[16] In the Supplementary Appendix S.B, we extend Álvarez (1988)'s analysis and provide conditions ensuring the concavity of value functions, as well as an example of an economy for which value functions are non-concave.

Theorem 1 ***A Necessary Condition for $\mathcal{A}-$efficiency.*** *Irrespective of the criterion used to identify potential agents, every $\mathcal{A}-$efficient allocation $\widehat{\mathsf{a}}$ satisfies, for each $t \geq 1$*

$$\lim_{\substack{i_{t+1} \longrightarrow \hat{i}^y_{t+1}(i^t) \\ i_{t+1} \in \mathsf{D}_{t+1}(i^t)}} \left(\frac{\int_{i_{t+1}}^{\hat{i}^y_{t+1}(i^t)} \mathcal{U}^D_{t+1}(\widehat{\mathsf{a}}; i^t, i) di}{\hat{i}^y_{t+1}(i^t) - i_{t+1}} \right) = \lim_{\substack{i_{t+1} \longrightarrow \hat{i}^y_{t+1}(i^t) \\ i_{t+1} \in \mathsf{D}_{t+1}(i^t)}} \overline{\mathcal{V}}^D_{t+1} \left(\frac{\int_{i_{t+1}}^{\hat{i}^y_{t+1}(i^t)} \mathsf{e}_{t+1}(i^t, i) di}{\hat{i}^y_{t+1}(i^t) - i_{t+1}} \right), \quad (20)$$

which, in environments in which $\mathcal{V}^D_{t+1}$ is concave, yields

$$\lim_{\substack{i_{t+1} \longrightarrow \hat{i}^y_{t+1}(i^t) \\ i_{t+1} \in \mathsf{D}_{t+1}(i^t)}} \mathcal{U}^D_{t+1}(\widehat{\mathsf{a}}; i^t, i_{t+1}) = \lim_{\substack{i_{t+1} \longrightarrow \hat{i}^y_{t+1}(i^t) \\ i_{t+1} \in \mathsf{D}_{t+1}(i^t)}} \mathcal{V}^D_{t+1} \left(\mathsf{e}_{t+1}(i^t, i_{t+1}) \right). \quad (21)$$

The intuition behind Theorem 1 is the following: if marginal children –that is, the youngest individuals in each family– living in a given allocation a do not use their income to maximize the utility of the dynasty head, she can improve on a by having more children that do use their endowment to maximize her utility. Since these newborn children require a lower income to provide the dynasty head with at least the same utility, they are, in a sense, "cheaper" than those children already living in a.

To understand the role played by the function $\overline{\mathcal{V}}^D_{t+1}$ in Theorem 1, a brief remark is in order. For an arbitrary agent i^t, write $e^y_{t+1}(i^t)$ for the limit of the average income available to the youngest children of i^t as i_{t+1} approaches $\hat{i}^y_{t+1}(i^t)$, that is,

$$\lim_{\substack{i_{t+1} \longrightarrow \hat{i}^y_{t+1}(i^t) \\ i_{t+1} \in \mathsf{D}_{t+1}(i^t)}} \frac{\int_{i_{t+1}}^{\hat{i}^y_{t+1}(i^t)} \mathsf{e}_{t+1}(i^t, i) di}{\hat{i}^y_{t+1}(i^t) - i_{t+1}} = e^y_{t+1}(i^t).$$

Recall that $\overline{\mathcal{V}}^D_{t+1}(e_{t+1})$ represents the maximum utility that the dynasty head can obtain by providing a set D_{t+1} –of positive measure– of her descendants with an average income e_{t+1}, and observe also that the function $\overline{\mathcal{V}}^D_{t+1}$ coincides with the function $\mathcal{V}^D_{t+1}$ if the latter is concave, but not in general. Moreover, when they do not coincide at a given point e_{t+1}, achieving $\overline{\mathcal{V}}^D_{t+1}(e_{t+1}) > \mathcal{V}^D_{t+1}(e_{t+1})$ is feasible because the dynasty head can select randomly the income available to each of her descendants in D_{t+1}. By Theorem 1, the average utility obtained from the youngest descendants of i^t must converge to $\overline{\mathcal{V}}^D_{t+1}(e^y_t(i^t))$. Therefore, if $\overline{\mathcal{V}}^D_{t+1}(e^y_t(i^t)) > \mathcal{V}^D_{t+1}(e^y_t(i^t))$ holds, then achieving $\mathcal{A}-$efficiency may require that the income available to the youngest descendants is determined randomly. Thus, non-convexities in the feasible set associated to endogenous fertility may cause that the only efficient allocations are possibly stochastic, a result also present (for Pareto efficiency) in Rogerson (1988).

Another implication of Theorem 1 is that, if an $\mathcal{A}-$efficient allocation is symmetric (in an *ex-ante* or *ex-post sense*) , then all descendants of the dynasty head must use their entire income to maximize the utility of the dynasty head. Thus, the utility obtained by the dynasty head with an $\mathcal{A}-$efficient allocation a can be written as

$$\mathcal{U}_0(\mathsf{a}; i^0) = W_0(\overline{e}_0, e_1, \overline{\mathcal{V}}^D_1(e_1)),$$

with

$$e_1 = \frac{1}{\mathsf{n}_1(i^0)} \int_{\mathsf{D}_1(i^0)} e_1(i)di = \int_{\mathbb{R}_+} e dE_1^{\mathsf{e}}(e, i^0).$$

Taking this into account and proceeding as in the two period economy of Section 2, it is straightforward to show that the average income received by the dynasty head's immediate descendants in every $\mathcal{A}-$efficient cannot be higher (or lower) than the average income received by the dynasty head's descendants in a dynastic optimum. That is, irrespectively of the criterion used to identify potential agents, every symmetric, $\mathcal{A}-$efficient allocation must be a dynastic optimum, and all Millian efficient allocations that are not dynastic optima are $\mathcal{A}-$inefficient. To save on space, we shall not prove this claim here.

Corollary 1 below states another implication of Theorem 1, arising in environments without altruism or with finite horizon altruism. In both environments, each function $\mathcal{V}_t^D$ is concave and the path $\{e_\tau\}_{\tau \geq t+1}$ solving the optimization problem in the definition of $\mathcal{V}_t^D(e_t)$ satisfies $e_\tau = 0$ for $\tau \geq t+1$. By Theorem 1, for each agent i^t alive at any given time, there must be a positive measure of agents whose income and utility are arbitrarily close to the the income and utility obtained by the agent's marginal child $i_{t+1}^y(i^t)$. Therefore, all descendants of these agents must also receive an income close to that received by the marginal child's descendants, that is, close to zero.

Corollary 1 *Suppose that utility functions adopt the finite-horizon altruistic form* $U\left(x_t, u_{t+1}^D\right) = v(x_t) + \delta u_{t+1}^D$ *and* $U^D(x_t, u_{t+1}^D) = v(x_t)$*, with* $\delta \in [0,1]$ *and* v *being strictly increasing in* c_t^m *and* n_{t+1}*. In this environment, for every* $\mathcal{A}-$*efficient allocation* a*, every* $t \geq 2$ *for which* $\mu^t\left\{\mathsf{D}^t(i^0)\right\} > 0$ *holds and every* $\epsilon_t > 0$*, there exists a set* $D \subseteq \mathsf{D}^t(i^0)$ *of agents* alive *at* t *of strictly positive measure for whom*

$$0 \leq \mathsf{e}_t(i^t) < \epsilon_t \tag{22}$$

is satisfied.

Two more comments are in order. First of all, observe that Corollary 1 requires that v is strictly increasing in c_t^m and n_{t+1}. Without this assumption, the set of agents for whom (22) holds might not even be born. With this assumption, (22), parents exhibiting finite-horizon altruistic preferences would like their descendants to have some grandchildren, so that they can use the labour capacities of the latter to obtain resources not for themselves, but for their parents. Thus, condition (22) must hold for a positive measure of *alive* agents born after $t = 2$. Second of all, we should note that, in our benchmark model, each function $\mathcal{U}_t^D$ is defined recursively, and therefore the only environment that exhibits finite horizon altruism in our general setting is that in which agents care only about consumption decisions by their *immediate* descendants. However, Corollary 1 suggests that a similar result characterizes $\mathcal{A}-$efficiency in every model in which the agents care about the welfare of a *finite* number of generations of their descendants.

As the set of feasible fertility choices increases, the set of allocations arising as $\mathcal{A}-$efficient with the *Birth-Date criterion* decreases. Theorem 2 below applies to environments for which feasible fertility choices include the set defined by

$$\mathcal{D}^C \equiv \{[a,b] : 0 \leq a \leq b \leq \overline{n}\},$$

that is, to environments in which parents may delay the birth of their first child, but then are restricted to giving birth to their other descendants successively in time. In such environments, if the child-rearing period $[0, \overline{n}]$ is long enough to allow the dynasty head to replace all her living children in a dynastic optimum, n_1^*, then $\mathcal{A}-$efficiency can be characterized as dynastic maximization.

Theorem 2 ***$\mathcal{A}-$efficiency as dynastic maximization.*** *Assume that $\mathcal{D}^C \subseteq \mathcal{D}$ and $n_1^* < \frac{\overline{n}}{2}$ hold. In this setting, an allocation a^* is $\mathcal{A}-$efficient –using the Birth-Date criterion to identify potential agents– if, and only if, it is a dynastic optimum.*

The intuition behind Theorem 2 is simple: the necessary condition in Theorem 1 implies that the number of children corresponding to an $\mathcal{A}-$efficient allocation cannot be higher than the number of children of the first generation, n_1^* , that the dynasty head would choose as optimum. Thus, if $2n_1^* < \overline{n}$ is satisfied, then any allocation a, which is not a dynastic optimum, is always $\mathcal{A}-$dominated by a dynastic optimum with a different set of living agents maximizing the utility of the dynasty head.

In environments with no altruism or finite horizon altruism, a straightforward implication of Theorem 2 is that every $\mathcal{A}-$efficient allocation must collapse in finite time. We state this result formally in Corollary 2 below.

Corollary 2 *Assume that $\mathcal{D}^C \subseteq \mathcal{D}$ and $n_1^* < \frac{\overline{n}}{2}$ hold. If, in addition, preferences exhibit no altruism or finite-horizon altruism, then every $\mathcal{A}-$efficient allocation a^* must satisfy $\mathsf{e}_t^*(i^t) = 0$ and $\mathsf{x}_t^*(i^t) = 0$ for each $t \geq 2$ and almost all $i^t \in \mathsf{D}^t(i^0)$. That is, in every $\mathcal{A}-$efficient allocation, the economy collapses in finite time.*

To summarize the results in this section, the notion of $\mathcal{A}-$efficiency is sensitive to the criterion by which potential lives are distinguished from one another. To be more precise, identifying potential agents by the date at which they may be born, rather than by the agents' position in their siblings birth order, reduces the set of allocations that can be regarded as $\mathcal{A}-$efficient. Yet, even the weakest form of $\mathcal{A}-$efficiency might be too demanding in environments with finite-horizon altruism, as it requires that some of the agents obtain an arbitrarily low income.

5 Millian efficiency as Robust $\mathcal{P}-$efficiency

In the general setting, $\mathcal{P}-$efficiency is associated to a sequence of functions $\mathcal{U}^N = \left\{\mathcal{U}_t : \mathcal{F} \times [0, \overline{n}]^t \longrightarrow \mathbb{R}^*\right\}$ determining the utility attributed to the unborn. Then, for any t and any potential agent of generation t, let $\mathcal{U}_t^P : \mathcal{F} \times [0, \overline{n}]^t \longrightarrow \mathbb{R}^*$ be defined, for all (a, i^t), by

$$\mathcal{U}_t^P(\mathsf{a}; i^t) = \begin{cases} \mathcal{U}_t(\mathsf{a}; i^t), & \text{if } i^t \in \mathsf{D}^t(i^0); \\ \mathcal{U}_t^N(\mathsf{a}; i^t), & \text{otherwise.} \end{cases}$$

With this notation, an allocation a $\mathcal{P}-$dominates an allocation a' if for every t and every $i^t \in [0, \overline{n}]^t$ one has $\mathcal{U}_t^P(\mathsf{a}; i^t) \geq \mathcal{U}_t^P(\mathsf{a}'; i^t)$, and there exists a period τ and a set of individuals $D^t \in \mathcal{B}\left[0, \overline{n}\right]^t$ of positive measure for which $\mathcal{U}_\tau^P(\mathsf{a}; i^\tau) > \mathcal{U}_\tau^P(\mathsf{a}'; i^\tau)$ is satisfied for all $i^\tau \in D^t$.

In their applications, GJT restrict the use of the term $\mathcal{P}-$dominance to a particular specification of the utilities attributed to the unborn for which the utility attributed to the

unborn is constant, that is, $\mathcal{U}_t^N(\mathsf{a}; i^t) = \overline{u}$, and their conclusions are analogous to those discussed in the two period economy discussed in Section 2. The main problem is that determining whether or not an allocation is optimal (*i.e.*, $\mathcal{P}-$efficient) becomes heavily dependent on the specific value attributed to the unborn, an unknown number.

There are, however, other possibilities. For example, the utility level attributed to an unborn agent might correspond to the lowest utility obtained by the agent's living siblings, that is, the utility function attributed to a potential agent $i^t = (i^{t-1}, i_t)$, if unborn, adopts the form

$$\mathcal{U}_t^N(\mathsf{a}; i^{t-1}, i_t) = \begin{cases} \inf\left\{\mathcal{U}_t(\mathsf{a}; i^{t-1}, i) : i \in \mathsf{D}_t(i^{t-1})\right\}, & \text{if } \mathsf{n}_t(i^{t-1}) > 0; \\ \inf\left\{\mathcal{U}_t(\mathsf{a}; i^t) : i^t \in \mathsf{D}^t(i^0)\right\}, & \text{if } \mathsf{n}_t(i^{t-1}) = 0. \end{cases} \tag{23}$$

Alternatively, we might assume that the utility attributed to an unborn agent is the average utility obtained by their living siblings, that is,

$$\mathcal{U}_t^N(\mathsf{a}; i^{t-1}, i_t) = \begin{cases} \frac{1}{\mathsf{n}_t(i^{t-1})}\left(\int_{\mathsf{D}_t(i^{t-1})} \mathcal{U}_t(\mathsf{a}; i^{t-1}, i)di\right), & \text{if } \mathsf{n}_t(i^{t-1}) > 0; \\ \frac{1}{\mu^t\{\mathsf{D}^t(i^0)\}}\left(\int_{\mathsf{D}^t(i^0)} \mathcal{U}_t(\mathsf{a}; i^{t-1}, i)di\right), & \text{if } \mathsf{n}_t(i^{t-1}) = 0. \end{cases} \tag{24}$$

With such specifications of the utility attributed to the unborn, an allocation a' with more individuals than an allocation a might not $\mathcal{P}-$dominate the allocation a even though all individuals living in both a and a' are better off in the new allocation a'.

The specifications of the utility for the unborn given in (23) or (24) might not represent the true preferences of potential agents and are, therefore, questionable. But any specification is, in our view, equally questionable. This is the initial stumbling block for any attempt to extend the Pareto criterion to compare allocations with different population size: there is no way of knowing whether or not a potential agent would be willing to be alive in any given allocation. Thus, rather than as assumptions on the utility obtained by non-born agents, we regard these specifications as a means to represent normative principles that determine under what conditions increasing or reducing the population size increases aggregate welfare. For example, according to the principle underlying the notion of $\mathcal{P}-$dominance arising when each function $\mathcal{U}_t^N$ is defined as in (23), a new life increases aggregate welfare only when the agent is not worse off than any of his/her living siblings with the same tastes and capacities. After all, one might argue that, when applied with the *Birth-Date criterion*, the notion of $\mathcal{A}-$dominance also represents a questionable normative principle, according to which a potential life should be worth living if the dynasty head decides that it is.

We should also emphasize that the $\mathcal{P}-$dominance criterion arising, for example, when each function $\mathcal{U}_t^N$ is defined as in (23), does not state that replacing an allocation a by an allocation a' with more individuals obtaining lower utility than any of their living siblings *decreases* aggregate welfare. If the dynasty head is better-off with the allocation a', then a and a' become non-comparable with the $\mathcal{P}-$dominance criterion. Despite the analogies of the functions defined in (23) or (24) with well-known social welfare functions, as an extension of the Pareto criterion, the $\mathcal{P}-$dominance criterion associated to each of these functions says nothing about whether or not any two living agents should redistribute their resources. Thus, basing the $\mathcal{P}-$dominance criterion on the specification given in (23) does not imply that a $\mathcal{P}-$efficient allocation must be symmetric, and using (24) does not mean

that a $\mathcal{P}-$efficient allocation must maximize the average welfare obtained by the living agents. In fact, $\mathcal{A}-$efficient allocations –that, as we have seen in the previous sections, are not necessarily symmetric– are always $\mathcal{P}-$efficient, and there are many other non-symmetric allocations that may be $\mathcal{P}-$efficient but are not $\mathcal{A}-$efficient.

It is true that, while the Pareto criterion does not introduce distributive concerns at all, when complemented with either (23) or (24), the notion of $\mathcal{P}-$dominance does introduce weak distributive concerns to determine under what conditions altering the population size is better from the point of view of aggregate welfare. And why shouldn't it? It would surely be odd to reject a reallocation of resources involving a unanimous welfare gain because some of the agents would envy those most favored by such a reallocation, *when the agents themselves have the opportunity to accept or reject it.* But, again, one cannot be sure that an increase in population size that makes the newborn envy all their living siblings is a true welfare improvement as the newborn will never have the opportunity to accept or reject that decision.

In the context of the two-period example in Section 2, it is easy to show that *i*) for each of the functions determining the utility of the unborn defined in (23) or (24), the $\mathcal{P}-$dominance criterion rules out many allocations as being $\mathcal{P}-$inefficient; *ii*) the $\mathcal{P}-$efficiency of a given allocation depends on whether potential agents are identified with the *Birth-Order criterion* or with the *Birth-Date criterion*; and *iii*) the $\mathcal{P}-$efficiency of an allocation depends, in general, of the utility function determining the utility of the unborn. Yet, Millian efficient allocations arise as $\mathcal{P}$-efficient independently of the criterion used to distinguish among potential agents and also of the function determining the utility of the unborn. That is, the $\mathcal{P}-$efficiency of Millian efficient allocations holds no matter the principle determining whether altering the population size increases aggregate is that captured by (23) or that captured by (24). In fact, in the context of the example, the $\mathcal{P}-$efficiency of Millian efficiency holds if the utility attributed to the unborn coincides with the median, the q^{th} quantile, or the average utility obtained by the agent's *elder* siblings. To summarize, one might take, to determine the utility attributed to the unborn, any symmetric function of the utility obtained by the agent's living siblings. Formally, in the context of the example, a sufficient condition to ensure the $\mathcal{P}-$efficiency of a Millian efficient allocation is that each function $\mathcal{U}_t^N$ satisfies the following property:

Property *S*. *For every* $t \geq 1$, *every* $i^t \in [0, \overline{n}]$ *and every ex-post symmetric allocation* a *such that* $\mathsf{x}_t(i^t) = x_t$ *and* $\mathsf{n}_{t+1}(i^t) = n_{t+1}$ *one has*

$$\mathcal{U}_t^N(\mathsf{a}; i^t) = \mathcal{U}_t(\mathsf{a}; i^t) \equiv U_t(a).$$

Theorem 3 establishes the equivalence between Millian efficiency and symmetric $\mathcal{P}-$efficiency in the context of the general setting described in Section 3. This equivalence requires a weak (concavity) condition. Given a sequence $\widehat{e} \equiv \{\widehat{e}_t\}_{t=0}^{\infty}$, for an arbitrary $t \geq 0$ and each e_t, let the restricted value function $\mathcal{V}_{\widehat{e},t}^D : [\widehat{e}_t, +\infty) \longrightarrow \mathbb{R}$ be defined, for every $e_t \in [\widehat{e}_t, +\infty)$, by

$$\mathcal{V}_{\widehat{e},t}^D(e_t) := \max_{i^t \in [0,\overline{n}]^t} \left\{ \max_{\mathsf{a}_t \in \mathcal{F}(e_t; i^t)} \left\{ \mathcal{U}_t^D(\mathsf{a}_t; i^t) : \mathsf{e}_\tau(i^\tau) \geq \widehat{e}_\tau \text{ for all } \tau \geq t+1 \right\} \right\}.$$

Note that, in contrast with the unconstrained value function $\mathcal{V}_t^D$, the constrained value function $\mathcal{V}_{\widehat{e},t}^D$ determines the maximum utility that agents born before t can obtain from the

consumption decisions of their descendants, *provided* each of these descendants is endowed with at least $\widehat{e}_\tau$ units of resources.

Theorem 3 ***A characterization of symmetric, $\mathcal{P}$-efficient allocations as Millian efficient allocations****. Assume Property* $\boldsymbol{S}$ *holds.*
i) If $\widehat{a}$ is a symmetric, $\mathcal{P}-$efficient allocation, then $\widehat{a}$ is Millian efficient;
ii) If $\widehat{a}$ is a Millian efficient allocation and, for each t, the function $\mathcal{V}^D_{\widehat{e},t}$ is concave on $[\widehat{e}_t, +\infty)$, then $\widehat{a}$ is $\mathcal{P}-$efficient.

Observe that Theorem 3.*i*) holds without restrictions on the set of feasible fertility choices. Therefore, the Milllian efficiency of symmetric, $\mathcal{P}-$efficient allocations holds irrespectively of the criterion to identify potential agents. A characterization of *any* $\mathcal{P}-$efficient allocation seems subtler, since determining whether or not a non-symmetric allocation is $\mathcal{P}-$efficient depends on the specific functional form given to each function $\mathcal{U}^N_t$. Yet, Theorem 3 suffices to rule out, as being $\mathcal{P}-$inefficient, any symmetric allocation that is not $\mathcal{M}-$efficient (for example, Benthamite optima).

Theorem 3.*ii*) establishes that, in regular settings in which value functions are concave on a certain range, Millian efficient allocations are $\mathcal{P}-$efficient as long as each function $\mathcal{U}^N_t$ belongs to the class of functions satisfying Property $\boldsymbol{S}$. Thus, just as an $\mathcal{A}-$efficient allocation can be described as a $\mathcal{P}-$efficient allocation for which $\mathcal{P}-$efficiency holds irrespectively of the utility attributed to the unborn, Millian efficient allocations can be described as $\mathcal{P}-$efficient allocations for which $\mathcal{P}-$efficiency holds for a wide range of specifications of the utility attributed to the unborn and, therefore, for a wide range of principles to compare allocations with different population size.

$\mathcal{P}-$efficiency and property rights. In Section 2, we express our concerns regarding the possibility that $\mathcal{A}-$efficiency might be incompatible with individual rights. In the two period economy described in Section 2, this occurs when the amount of resources that children can obtain with the labour capacity they are endowed with, w, is higher than average income received by children in a dynastic optimum, e^*_1. Also, the allocation arising with said distribution of rights is Millian efficient and, in view of Theorem 3 above, $\mathcal{P}-$efficient. Therefore, as in those environments in which achieving Pareto-efficiency is incompatible with a distribution of rights and achieving Pareto efficiency from an initial *status quo* is not Pareto improving –as occurs, for example, in environments with asymmetric information or in overlapping generations economies facing macroeconomic risks–, it might be convenient to adopt weaker notions of efficiency such as $\mathcal{P}-$efficiency, just as it is useful to adopt interim (incentive-constrained) efficiency[17] in the presence of asymmetric information or interim –or conditional– efficiency in stochastic, overlapping generations economies (see, e.g., Chattopaday and Gotardi, 1999).

In the latter case, achieving Pareto efficiency (*ex-ante* efficiency in this case) from an initial *status quo* requires policies –social security– that make a generation of agents worse-off than they would be in the absence of such policies, which may, therefore, be time inconsistent. In our example, an analogous problem may occur. There, achieving $\mathcal{A}-$efficiency (or, with the *Birth-Date criterion*, dynastic maximization) requires that the dynasty head accumulates debts that their descendants may not be willing to repay.

[17] See Holmström and Myerson (1981).

In the Supplementary Appendix S.C, we explore, in the context of the general setting in Section 3, the efficiency properties of a decentralized mechanism in which the agents, endowed with well defined property rights on their labour capacities, trade in competitive markets and are allowed to voluntary transfer resources to their descendants. While the equilibria arising from the interaction of markets and families can only be $\mathcal{A}-$ efficient if 1) preferences exhibit dynastic altruism and 2) the non-negativity constraint on gifts and bequests is never binding, it may be $\mathcal{P}-$efficient under much more general circumstances. In Theorem S.C.1 we show that, when equilibria are symmetric and prices satisfy an appropriate transversality –or dynamic efficiency-condition, competitive equilibria are Millian efficient and, hence, $\mathcal{P}-$efficient. Thus, policies pursuing $\mathcal{A}-$efficiency by altering these equilibria may be time inconsistent.

6 Conclusions

In this paper, we have explored the properties of the notions of $\mathcal{A}-$efficiency and $\mathcal{P}-$efficiency, proposed by Golosov, Jones and Tertilt (2007), to evaluate allocations in a general overlapping generations setting with endogenous fertility and descendant altruism. We have first argued that achieving $\mathcal{A}-$efficiency may have different implications depending on the criterion we use to distinguish potential agents from one another. If we identify potential siblings by their birth order, the set of $\mathcal{A}-$efficient allocations is large, although the youngest siblings in every family must devote most of their income to maximize the utility of their parents. Therefore, $\mathcal{A}-$efficiency might be in conflict with the individuals' rights to use their labour capacity as they wish; and, in environments with finite horizon altruism, it implies that the youngest children in almost every family obtain almost zero resources to finance their own consumption and fertility plans. Things might get worse –for everyone except the dynasty head– if we identify potential agents by their birth dates. In this case, achieving $\mathcal{A}-$efficiency requires that all children –that, in our setting, are equal in their tastes and capacities– are treated symmetrically and therefore, it requires that all agents in the economy must devote their entire income to maximize the utility of their parents. Since, in our setting, the preferences of any agent on consumption decisions of his/her descendants coincide with those of the dynasty head, $\mathcal{A}-$efficiency reduces, therefore, to dynastic maximization. In environments with finite horizon altruism, for example, $\mathcal{A}-$efficiency is characterized by a collapse.

In the paper, we have also shown that these properties might not hold for $\mathcal{P}-$efficiency. At least, if the welfare attributed to the unborn –which we regard as a device that represents different principles determining under what circumstances new lives increase welfare– depends on the welfare enjoyed by those living in a given allocation. More specifically, if the welfare attributed to the unborn is a symmetric function of the welfare obtained by their living siblings, then every Millian efficient allocation –that is, every symmetric allocation that is not $\mathcal{A}-$dominated by any other symmetric allocation–, is $\mathcal{P}-$efficient. Finally, we have provided a version of the First Welfare Theorem by showing that *i*) every symmetric competitive equilibrium is a —statically– Millian efficient allocation; and, that *ii*) if the equilibrium is dynamically efficient, then it is both Millian efficient and $\mathcal{P}-$efficient. Thus, an important qualitative conclusion of Golosov, Jones and Tertilt prevails: in the absence of non convexities, externalities, missing markets, dynamic efficiency problems, etc., the fact

that fertility decisions are endogenous does not mean that markets fail to deliver efficient allocations.

There are several directions that might be worth exploring. A first direction would be to extend the results to environments in which the agents are heterogeneous. While an environment populated by agents with identical preferences exhibiting no altruism –or finite horizon altruism– are probably extremely rare, an environment populated by heterogeneous agents in which some of the agents' preferences have these properties are probably less rare. Thus, Corollary 1 may become more relevant in models with heterogeneous dynasties. We should also point out that the symmetry restriction underlying the Millian notion of efficiency requires that every two agents with the same characteristics are treated equally, but this does not mean that agents with different characteristics are treated equally. Thus, in models in which agents are heterogeneous in their characteristics (preferences, endowments, preferences and endowments of their ancestors and, finally, the agents' birth dates or the agents' order of birth with respect to their siblings), the Millian notion of efficiency may be still applicable if we regard the symmetry restriction as requiring that any two agents of the same generation with the same preferences and endowments –and for whom the preferences and endowments of all their ancestors are also equal– must be treated symmetrically. If the utility attributed to an unborn agent depends only on the utility attributed to those among the agent's siblings that are equal, then the equivalence between Millian efficiency and symmetric $\mathcal{P}-$efficiency will prevail. As in the setting studied here (in which many asymmetric allocations might be regarded as $\mathcal{P}-$efficient even when they are not $\mathcal{A}-$efficient), $\mathcal{P}-$efficiency is consistent with symmetry but it does not impose symmetry.

A second direction would be to extend the results to a –perhaps– more realistic model of fertility choice in which children are born at different periods of time –with at most one child being born in any period– that, differently from the setting studied here, may be relevant from the point of view of parents' preferences. Note that, in this case, $\mathcal{A}-$efficiency might differ from dynastic maximization even if one uses the *Birth-Date criterion* to identify potential agents. Yet, achieving $\mathcal{A}-$efficiency might also be in conflict with children's rights, especially if some of the agents exhibit finite-horizon altruism. In such a model, the date of birth of potential agents becomes relevant for their parents, and using $\mathcal{P}-$efficiency as we propose here might involve some difficulties.

Finally, as a third direction, it might be worthwhile to explore the consequences of different fertility policies in environments in which other potential market failures arise, such as pollution problems, missing markets, etcetera.

Appendix: Proofs

Proof of Theorem 1. Let $\widehat{\mathsf{a}}$ be an $\mathcal{A}-$efficient allocation. In order to show that (20) must be satisfied, assume first that, for the allocation $\widehat{\mathsf{a}}$, all fertility choices chosen at t belong to the set D^0, so that $\widehat{\imath}^y_{t+1}(i^t) = \widehat{n}_{t+1}(i^t)$ is satisfied. Note that this assumption is without loss of generality: since the agents do not care about the specific points in time at which their children are born, for every $\mathcal{A}-$efficient allocation, there must be an allocation whose fertility choices satisfy this property that is also $\mathcal{A}-$efficient.

Write now $\widehat{x}_t$ and $\widehat{k}_{t+1}$, respectively, for $\widehat{x}_t = \widehat{\mathsf{x}}_t(i^t)$ and $\widehat{k}_{t+1} = \widehat{\mathsf{k}}^o_{t+1}(i^t)$. Also, for each $n \in [0, \overline{n}]$, write $e^{\widehat{\mathsf{a}}}_{t+1}(n)$ and $D^{\widehat{\mathsf{a}}}_{t+1}(n)$, respectively, for $e^{\widehat{\mathsf{a}}}_{t+1}(n) = \int_0^n \widehat{\mathsf{e}}_{t+1}(i^t, i)di$ and $D^{\widehat{\mathsf{a}}}_{t+1}(n) =$

$\int_0^n \mathcal{U}_{t+1}^D(\widehat{\mathsf{a}}; i^t, i)di$. With this notation, observe that the welfare obtained by the dynasty head by choosing a feasible allocation with $n \leq \widehat{n}_{t+1}$ individuals born at period $t+1$, if all descendants in the interval $[0, n]$ take the same decisions as those corresponding to the allocation $\widehat{\mathsf{a}}$, can also be written as

$$U_t^{D\widehat{\mathsf{a}}}(n) = U^D\left(\overline{e}_0 - b_t(n) - \widehat{k}_1^o, F_{t+1}(\widehat{k}_{t+1}^o, n) - e_{t+1}^{\widehat{\mathsf{a}}}(n), n, \frac{D_{t+1}^{\widehat{\mathsf{a}}}(n)}{n}\right) = \frac{D_t^{\widehat{\mathsf{a}}}(n)}{n}.$$

From the definition of $\mathcal{A}-$efficiency, any allocation differing from $\widehat{\mathsf{a}}$ at a single point –or, in general, on a set of measure zero– must be also $\mathcal{A}-$efficient. Therefore, since both $e_{t+1}^{\widehat{\mathsf{a}}}$ and $D_{t+1}^{D\widehat{\mathsf{a}}}$ are integrable and the Lebesgue integrals $e_{t+1}^{\widehat{\mathsf{a}}}$ and $D_{t+1}^{D\widehat{\mathsf{a}}}$ are differentiable almost everywhere,[18] there is no loss of generality in assuming that both functions are differentiable from the left –and, hence, continuous from the left–, at $\widehat{n}_{t+1}$. That is,

$$\begin{aligned}
\frac{d^- e_{t+1}^{\widehat{\mathsf{a}}}(\widehat{n}_{t+1})}{dn} &= \lim_{\substack{n\to\widehat{n}_{t+1}\\ n<\widehat{n}_{t+1}}} \frac{\int_n^{\widehat{n}_{t+1}} \widehat{\mathsf{e}}_{t+1}(i^t, i)di}{\widehat{n}_{t+1} - n} = \widehat{\mathsf{e}}_{t+1}(i^t, \widehat{n}_{t+1});\\
\frac{d^- D_{t+1}^{\widehat{\mathsf{a}}}(\widehat{n}_{t+1})}{dn} &= \lim_{\substack{n\to\widehat{n}_{t+1}\\ n<\widehat{n}_{t+1}}} \frac{\int_n^{\widehat{n}_{t+1}} \mathcal{U}_{t+1}^D(\widehat{\mathsf{a}}; i^t, i)di}{\widehat{n}_{t+1} - n} = \mathcal{U}_{t+1}^D(\widehat{\mathsf{a}}, i^t, \widehat{n}_{t+1}).
\end{aligned}$$

Moreover, since $\widehat{\mathsf{a}}$ is $\mathcal{A}-$efficient and the dynasty head cannot obtain higher utility by reducing the population size, the left-hand side derivative of $U_0^{\widehat{\mathsf{a}}}$ at $\widehat{n}_{t+1}$ satisfies

$$\begin{aligned}
\frac{d^- U_t^{D\widehat{\mathsf{a}}}(\widehat{n}_{t+1})}{dn_{t+1}} &= -b_t'(\widehat{n}_{t+1}) D_1 U^D\left(\widehat{x}_t, U_{t+1}^{D\widehat{\mathsf{a}}}(\widehat{n}_{t+1})\right) +\\
&\quad + \left[D_2 F_{t+1}(\widehat{k}_{t+1}^o, \widehat{n}_{t+1}) - \widehat{\mathsf{e}}_{t+1}(i^t, \widehat{n}_{t+1})\right] D_2 U^D\left(\widehat{x}_t, U_{t+1}^{D\widehat{\mathsf{a}}}(\widehat{n}_{t+1})\right) +\\
&\quad + D_3 U^D\left(\widehat{x}, U_{t+1}^{D\widehat{\mathsf{a}}}(\widehat{n}_{t+1})\right) +\\
&\quad + \frac{1}{\widehat{n}_{t+1}}\left[\mathcal{U}_{t+1}^D(\widehat{\mathsf{a}}, i^t, \widehat{n}_{t+1}) - U_{t+1}^{D\widehat{\mathsf{a}}}(\widehat{n}_{t+1})\right] D_4 U^D\left(\widehat{x}, U_{t+1}^{D\widehat{\mathsf{a}}}(\widehat{n}_{t+1})\right)\\
&\geq 0.
\end{aligned}$$

With this observation in mind, we now show that condition (20), that is,

$$\lim_{\substack{n\to\widehat{n}_{t+1}\\ n<\widehat{n}_{t+1}}} \left(\frac{\int_n^{\widehat{n}_{t+1}} \mathcal{U}_{t+1}^D(\widehat{\mathsf{a}}; i^t, i)di}{\widehat{n}_{t+1} - n}\right) = \overline{\mathcal{V}}_{t+1}^D\left(\lim_{\substack{n\to\widehat{n}_{t+1}\\ n<\widehat{n}_{t+1}}} \frac{\int_n^{\widehat{n}_{t+1}} \widehat{\mathsf{e}}_{t+1}(i^t, i)di}{\widehat{n}_{t+1} - n}\right) = \overline{\mathcal{V}}_{t+1}^D\left(\widehat{\mathsf{e}}_{t+1}(i^t, \widehat{n}_{t+1})\right) \quad (25)$$

must be satisfied. To prove this statement, suppose it is false. Then select $\widetilde{e}_{t+1} < \widehat{\mathsf{e}}_{t+1}(i^t, \widehat{n}_{t+1})$ in such a way that

$$\lim_{\substack{n\to\widehat{n}_{t+1}\\ n<\widehat{n}_{t+1}}} \left(\frac{\int_n^{\widehat{n}_{t+1}} \mathcal{U}_{t+1}^D(\widehat{\mathsf{a}}; i^t, i)di}{\widehat{n}_{t+1} - n}\right) = \overline{\mathcal{V}}_{t+1}^D(\widetilde{e}_{t+1})$$

is satisfied and consider an allocation a such that: *i*) at time t, agent i^t chooses $\widetilde{n}_{t+1} > \widehat{n}_{t+1}$; *ii*) those individuals who were already living in $\widehat{\mathsf{a}}$ receive exactly the same bundle; and, *iii*) those individuals born at t who were not living under $\widehat{\mathsf{a}}$ receive an endowment $e_{t+1}(i^t, i_t) = \widetilde{e}_{t+1}$ and use

[18] See, e.g., Theorem 10 in Royden (1988, p.107).

this endowment to maximize the utility of the dynasty head, so that $e^{\widehat{\mathsf{a}}}_{t+1}(n) = \int_0^n \widehat{\mathsf{e}}_{t+1}(i^t, i)di + [\widetilde{n}_{t+1} - n]\widetilde{e}_{t+1}$ is satisfied. Since the number of individuals alive in the new allocation a is higher than the number of individuals living in $\widehat{\mathsf{a}}$, the left-hand side derivative of $U_t^{D,\mathsf{a}}$ at $n = \widehat{n}_{t+1}$ coincides with that of $U_{t+1}^{D\widehat{\mathsf{a}}}$. Also, the right-hand side derivative of $U_{t+1}^{D,\mathsf{a}}$ at $n = \widehat{n}_{t+1}$ is given by

$$\frac{d^+U_t^{D\widehat{\mathsf{a}}}(\widehat{n}_{t+1})}{dn_{t+1}} = \frac{d^-U_t^{D\widehat{\mathsf{a}}}(\widehat{n}_{t+1})}{dn_{t+1}} + D_2U^D\left(\widehat{x}, U_{t+1}^{D\widehat{\mathsf{a}}}(\widehat{n}_{t+1})\right)\left[\widehat{\mathsf{e}}_{t+1}(i^t, \widehat{n}_{t+1}) - \widetilde{e}_{t+1}\right] > 0,$$

which implies that there exists $\widetilde{n}_{t+1} > \widehat{n}_{t+1}$ for which the dynasty head –and, under assumption $A2$, agent i^t–obtains more utility with the allocation a than the utility she obtains with $\widehat{\mathsf{a}}$. Thus, some of the agents living in both a and $\widehat{\mathsf{a}}$ are better off with the former allocation than they are with the latter, and no agent living in the two allocations is worse off. This contradicts the assumption imposing that $\widehat{\mathsf{a}}$ is $\mathcal{A}$–efficient, a contradiction that establishes that (25) –and, hence, the equivalent condition in (20)– must be satisfied. Note that, if the value function $\mathcal{V}_{t+1}^D$ is concave and, hence, satisfies $\mathcal{V}_{t+1}^D = \overline{\mathcal{V}}_{t+1}^D$; then (21) follows straightforwardly from (20), which completes the proof of Theorem 1. ■

Proof of Corollary 1. Let a be an $\mathcal{A}$–efficient allocation arising in an environment with no altruism or with finite horizon altruism, and let $t \geq 1$ and i^t be arbitrary. As in the proof of Theorem 1, we shall assume, without loss of generality, that $\mathcal{D} \equiv \mathcal{D}^O$ holds. As in this type of environments, $U_t^D(\mathsf{a}, i^t) = u(x_t(i^t))$ is satisfied, it is straightforward to show that the function $\mathcal{V}_t^D$ can be equivalently defined, for each $e_t \geq 0$, by

$$\mathcal{V}_t^D(e_t) = \max\left\{W_t(e_t, e_{t+1}) : e_{t+1} \geq 0\right\};$$

where, in turn, $W_t : \mathbb{R}_+^2 \to \mathbb{R}$ is defined, for each $(e_t, e_{t+1}) \in \mathbb{R}_+^2$, by

$$W_t(e_t, e_{t+1}) = \max_{(k_{t+1}^o, x_{t+1}) \in \mathbb{R}_+^3 \times [0,\overline{n}]} \left\{u(x_t) : c_t^m + b_0(n_1) + k_1^o \leq \overline{e}_t;\ c_1^o + n_{t+1}e_{t+1} \leq F_1(k_1^o, n_1)\right\}.$$

It is straightforward to show that W_t is strictly increasing in e_t and strictly decreasing in e_{t+1}, so that $\mathcal{V}_t^D$ is a concave function satisfying $\mathcal{V}_t^D(e_t) = W_t(e_t, 0)$.

On the other hand, as the agents care only on consumption-fertility decisions of their immediate descendants, it is also straightforward to show that, in every $\mathcal{A}$–efficient allocation, for each $t \geq 1$ and each i^t alive at t, i^t (and, by Assumption $A1$, the dynasty head as well) must choose $(\mathsf{x}_t(i^t), \mathsf{k}_{t+1}^o(i^t))$ to solve the optimization problem in the definition of $W_t(e_t, \int_{\mathbb{R}_+} e dE_{t+1}^{\mathsf{e}}(e, i^t))$, which yields

$$\mathcal{U}_t^D(\mathsf{a}, i^t) = W_t\left(e_t, \int_{\mathbb{R}_+} e dE_{t+1}^{\mathsf{e}}(e, i^t)\right).$$

Therefore, Theorem 1 implies that

$$\lim_{\substack{i_{t+1} \to \mathsf{n}_{t+1}(i_t) \\ i_{t+1} < \mathsf{n}_{t+1}(i_t)}} W_{t+1}\left(e_{t+1}(i^t, i_{t+1}), \int_{\mathbb{R}_+} e dE_{t+1}^{e}(e, i^t, i_{t+1})\right) = \lim_{\substack{i_{t+1} \to \mathsf{n}_{t+1}(i_t) \\ i_{t+1} < \mathsf{n}_{t+1}(i_t)}} W_{t+1}\left(e_{t+1}(i^t, i_{t+1}), 0\right). \tag{26}$$

must be satisfied. Observe that the fact that W_{t+1} is continuous, together with the fact the two limits in (26) exist imply that the limit

$$\lim_{\substack{i_{t+1} \to \mathsf{n}_{t+1}(i_t) \\ i_{t+1} < \mathsf{n}_{t+1}(i_t)}} e_{t+1}(i^t, i_{t+1})$$

is also well defined. Taking this into account, to complete the proof of Corollary 1, suppose it does not hold for, say, period $\tau = t+2$. That is, there exists $\epsilon_{t+2} > 0$ such that for (almost) every agent of generation $\tau = t+2$ alive in a one has $e_{t+1}(i^{t+1}, i_{t+2}) \geq \epsilon_{t+2}$. Observe that this implies that $\int_{\mathbb{R}_+} e dE^e_{t+1}(e, i^{t+1}) \geq \epsilon_{t+2}$ must be satisfied for (almost) every agent ι^t alive at $\tau = t+1$. But the fact that W_{t+1} is strictly decreasing in e_{t+2} implies that

$$\lim_{\substack{i_{t+1}\to \mathsf{n}_{t+1}(i_t)\\ i_{t+1}<\mathsf{n}_{t+1}(i_t)}} W_{t+1}\left(e_{t+1}(i^t, i_{t+1}), \int_{\mathbb{R}_+} e dE^e_{t+1}(e, i^t, i_{t+1})\right) < \lim_{\substack{i_{t+1}\to \mathsf{n}_{t+1}(i_t)\\ i_{t+1}<\mathsf{n}_{t+1}(i_t)}} W_{t+1}\left(e_{t+1}(i^t, i_{t+1}), 0\right) \tag{27}$$

must hold, which contradicts Theorem 1, a contradiction that completes the proof of Corollary 1. ■

Proof of Theorem 2. Let $\widehat{\mathsf{a}}$ be an $\mathcal{A}-$efficient allocation in an economy for which $\mathcal{D} \equiv \mathcal{D}^C$ and assume $2n_1^* < \overline{n}$ is satisfied. To prove Theorem 2, we proceed by steps: in a first step, we show that, in the allocation $\widehat{\mathsf{a}}$, the number of individuals obtaining an income above the average income e_1^* corresponding to a dynastic optimum is lower than the number of individuals n_1^* corresponding to such a dynastic optimum; in a second step, we show that, under the qualifying condition $2n_1^* < \overline{n}$, we must have $\widehat{\mathsf{n}}_1(i^0) = n_1^*$, which, in turn, implies that $\widehat{\mathsf{a}}$ must be a dynastic optimum.

Step 1. To prove Step 1, for each $t \geq 1$, each $i^t \in \widehat{\mathsf{D}}_t(i^0)$ and each $e \geq 0$, let $\mathcal{V}_t^{D,\widehat{\mathsf{a}}}(e_t, i^t)$ be defined as the maximal utility that agent i^t's parent –or, in models with infinite horizon altruism, the dynasty head– can obtain from i^t's descendants by endowing i^t with e_t units of resources, provided each descendant i^τ of i^t has to be provided with at least the same resources as the resources she receives in $\widehat{\mathsf{a}}$. That is,

$$\mathcal{V}_t^{D,\widehat{\mathsf{a}}}(e_t, i^t) := \max_{\mathsf{a}_t \in \mathcal{F}(e_t; i^t)} \left\{\mathcal{U}_t^D(\mathsf{a}; i^t) : \mathsf{e}_\tau(i^\tau) \geq \widehat{\mathsf{e}}_\tau(i^\tau) : \tau > t;\ i^\tau \in \mathsf{D}^\tau(i^0)\right\}. \tag{28}$$

With this notation, it is straightforward to show that, since $\widehat{\mathsf{a}}$ is $\mathcal{A}-$efficient, one must have, for every $t \geq 1$ and every $i^t \in \widehat{\mathsf{D}}^t(i^0)$,

$$\mathcal{U}_t^D(\widehat{\mathsf{a}}; i^t) = \mathcal{V}_t^{D,\widehat{\mathsf{a}}}(\widehat{\mathsf{e}}_t(i^t); i^t). \tag{29}$$

With this observation in mind, note that, for any allocation a arising as $\mathcal{A}-$efficient in this unrestricted setting, there is an allocation $\widehat{\mathsf{a}}$ satisfying $\widehat{\mathsf{D}}_{t+1}(i^t) = \left[0, \widehat{\mathsf{n}}_{t+1}(i^t)\right]$ for every $i^t \in \mathsf{D}^t(i^0)$ that provides the dynasty head with the same utility as the utility that she obtains with a, that, consequently, is also $\mathcal{A}-$efficient. Thus, we can assume, without loss of generality, that $\widehat{\mathsf{D}}_1(i^0)$ adopts the form $\left[0, \widehat{\mathsf{n}}_1(i^0)\right]$ .Taking this is this into account, the fact that $\widehat{\mathsf{a}}$ is $\mathcal{A}-$efficient implies that the utility obtained with $\widehat{\mathsf{a}}$ by the dynasty head can be written as

$$\begin{aligned}\mathcal{U}_0(\widehat{\mathsf{a}}; i^0) \quad = \quad & \max_{(e_1^y, k_1^o, x_0)\in\mathbb{R}^4_+\times[0,\overline{n}]} \left\{U\left(x_0, \frac{1}{n_1}\int_0^{\underline{n}_1} \mathcal{V}_1^{D,\widehat{\mathsf{a}}}(\widehat{\mathsf{e}}_1(i), i)di + (n_1 - \underline{n}_1)\,\overline{\mathcal{V}}_1^D(e_1^y)\right) :\right. \\ & \left. c_0^m + b_0(n_1) + k_1^o \leq \overline{e}_0;\ c_1^o + \int_0^{\underline{n}_1} \mathsf{e}_1(i)di + (n_1 - \underline{n}_1)\, e_1^y \leq F_1(k_1^o, n_1)\right\}\end{aligned}$$

for some $\underline{n}_1 \in \left[0, \widehat{\mathsf{n}}_1(i^0)\right]$. Note that if $\underline{n}_1 = 0$, the allocation $\widehat{\mathsf{a}}$ is a dynastic optimum. Therefore, for the remaining of the proof we shall assume that $\underline{n}_1 > 0$.

Write now $\widehat{e}_1^o$ and $\overline{\mathcal{V}}_1^{D,\widehat{\mathsf{a}},\underline{n}_1}(e_1^o)$, respectively, for $\widehat{e}_1^o = \frac{1}{\underline{n}_1}\int_0^{\underline{n}_1} \widehat{\mathsf{e}}_1(i)di$ and

$$\overline{\mathcal{V}}_1^{D,\widehat{\mathsf{a}},\underline{n}_1}(e_1^o) = \max_{e_1:[0,\underline{n}_1]\to R_+} \left\{\frac{1}{\underline{n}_1}\int_0^{\underline{n}_1} \mathcal{V}_1^{D,\widehat{\mathsf{a}}}(\mathsf{e}_1(i), i)di : \frac{1}{\underline{n}_1}\int_0^{\underline{n}_1} \mathsf{e}_1(i)di = e_1^o\right\}.$$

Using this notation, it can be shown that the utility obtained with $\widehat{\mathsf{a}}$ by the dynasty head can be written as

$$\mathcal{U}_0(\widehat{\mathsf{a}};i^0) = \max_{(e_1^o,e_1^y,k_1^o,x_0)\in\mathbb{R}_+^5\times[0,\overline{n}]} \left\{ U\left(x_0, \left(\frac{\underline{n}_1}{n_1}\right)\overline{\mathcal{V}}_1^{D,\widehat{\mathsf{a}},\underline{n}_1}(e_1) + \left(1-\frac{\underline{n}_1}{n_1}\right)\overline{\mathcal{V}}_1^{D}(e_1^y)\right) : \right. \tag{30}$$
$$\left. e_1 \geq \widehat{e}_1^o;\ c_0^m + b_0(n_1) + k_1^o \leq \overline{e}_0;\ c_1^o + \left(\frac{\underline{n}_1}{n_1}\right)e_1 + \left(1-\frac{\underline{n}_1}{n_1}\right)e_1^y \leq F_1(k_1^o,n_1)\right\}.$$

Using the first order conditions characterizing a dynastic optimum, together with those characterizing a solution $(\widehat{x}_0,\widehat{k}_1^o,\widehat{e}_1^o,\widehat{e}_1^y)$ to the optimization problem in (30) and taking into account that both $\overline{\mathcal{V}}_1^D$ and $\overline{\mathcal{V}}_1^{D,\widehat{\mathsf{a}},\underline{n}_1}$ are both concave –even if $\mathcal{V}_1^{D,\widehat{\mathsf{a}}}$ and $\mathcal{V}_1^D$ are not concave– it can be shown that $\widehat{e}_1^o \geq \widehat{e}_1^y$ and $\widehat{e}_1^o \geq e_1^*$ must be satisfied.

It can be shown –using the first-order conditions of (30) and those characterizing a dynastic optimum– that, if $\widehat{e}_1^o \geq \widehat{e}_1^y > e_1^*$, then $\widehat{n}_1 < n_1^*$ must be satisfied. Also, if $\widehat{e}_1^o \geq e_1^* > \widehat{e}_1^y$ is satisfied, then $\underline{n}_1 < n_1^*$ must be satisfied. Therefore, in the allocation $\widehat{\mathsf{a}}$, the number of individuals obtaining an income above the average income e_1^* corresponding to a dynastic optimum is lower than the number of individuals n_1^* corresponding to such a dynastic optimum, which completes the proof of Step1.

Step 2: $\widehat{\mathsf{n}}_1(i^0) = n_1^*$. Taking Step 1 into account, it is easy to see that $\widehat{\mathsf{a}}$ cannot be $\mathcal{A}-$efficient unless it is a dynastic optimum:. If it was $\mathcal{A}-$efficient but it was not a dynastic optimum, it would be easy to replace $\widehat{\mathsf{a}}$ by a dynastic optimum a^* such that most of the agents living in a^*, except the dynasty head and all those obtaining $\widehat{e}_1^y < e_1^*$, are not alive in $\widehat{\mathsf{a}}$. Such allocation a^* would trivially provide all agents living in both a^* and $\widehat{\mathsf{a}}$ with more utility than the utility that they obtain with $\widehat{\mathsf{a}}$, which contradicts the hypothesis stating that $\widehat{\mathsf{a}}$ is $\mathcal{A}-$efficient but it is not a dynastic optimum.

Thus, every $\mathcal{A}-$efficient allocation must be a dynastic optimum. Since dynastic optima are trivially $\mathcal{A}-$efficient, it follows that, in a setting in which $\mathcal{D} \subseteq \mathcal{D}^C$, an allocation is $\mathcal{A}-$efficient if, and only if, it is a dynastic optimum, which completes the proof of Theorem 2. ■

Proof of Corollary 2. Observe, that, under the qualifying conditions, $\mathcal{D} \subseteq \mathcal{D}^C$ and $n_1^* < \frac{\overline{n}}{2}$, every $\mathcal{A}-$efficient allocation a^* must be a dynastic optimum by Theorem 2. Also, proceeding as in the proof of Corollary 1, one must have $\mathcal{V}_t^D(e_t) = W_t(e_t,0)$ for each t. It follows that $e_t = 0$ must be satisfied for $t \geq 2$, which completes the proof. ■

Proof of Theorem 3. To prove i), assume Property $\boldsymbol{S}$ holds and let $\widehat{\mathsf{a}}$ be an *ex-post* symmetric, $\mathcal{P}-$efficient allocation satisfying $\widehat{\mathsf{x}}_t(i^t) = \widehat{x}_t$ for each t and each $i^t \in \mathsf{D}_t(i^0)$. To show that $\widehat{a}$ must be Millian efficient, suppose it is not; that is, suppose there exists an alternative symmetric allocation a that provides all generations of agents living in $\widehat{\mathsf{a}}$ with higher utility. Since Property $\boldsymbol{S}$ holds, choosing a instead of $\widehat{a}$ involves a welfare improvement from the point of view of the $\mathcal{P}-$dominance criterion, which contradicts the assumption imposing that $\widehat{a}$ is $\mathcal{P}-$efficient and, hence, completes the proof of the i) statement in Theorem 3.

To prove ii), assume Property $\boldsymbol{S}$ holds and let $\widehat{a}$ be a Millian efficient allocation such that each function $\mathcal{V}_{\widehat{e},t}^D$ is concave on $[\widehat{e}_t,+\infty)$. To show that $\widehat{a}$ is $\mathcal{P}-$efficient, let t be arbitrary and write $\mathcal{V}_{\widehat{e},t}^D(\widehat{e}_t)$ as

$$\mathcal{V}_{\widehat{e},t}^D(\widehat{e}_t) = \max_{E:[\widehat{e}_{t+1},\infty]\to[0,1]\in\Delta\mathbb{R}_+} W_t^D\left(\widehat{e}_t, \int dE(e), \int \mathcal{V}_{\widehat{e},t+1}^D(e)dE(e)\right),$$

which taking into account that $\mathcal{V}_{\widehat{e},t+1}^D$ is concave on $[\widehat{e}_{t+1},+\infty)$, implies that $\widehat{e}_{t+1}$ solves the sequence of optimization problem in the definition of $\left\{\mathcal{V}_{\widehat{e},t}^D(\widehat{e}_t)\right\}_{t\geq 1}$. Therefore, $\mathcal{U}_t^D(\widehat{\mathsf{a}},i^t) = \mathcal{V}_{\widehat{e},t}^D(\widehat{e}_t)$ must

be satisfied for each t and each $i^t \in \mathsf{D}^t(i^0)$. Analogously, $\mathcal{U}_0(\widehat{a}; i^0) = \mathcal{V}_{\widehat{e},0}(\overline{e}_0)$ must hold for $t = 0$. Taking this into account, observe that if $\widehat{a}$ is not $\mathcal{P}-$efficient, then it should be $\mathcal{P}-$dominated by the allocation solving the sequence of optimization problems in the definition of $\left\{\mathcal{V}^D_{\widehat{e},t}(\widehat{e}_t)\right\}_{t\geq 1}$, a contradiction that establishes that $\widehat{a}$ is $\mathcal{P}$-efficient and completes the proof of Theorem 3. ■

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Supplement to "*Efficiency and Endogenous Fertility*"

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Appendix S.A: The notion of Millian efficiency with endogenous fertility

In this Supplementary Appendix S.A. we characterize Millian efficient allocations.

A feasible allocation $\mathsf{a} \in \mathcal{F}$ is said to be *ex-post symmetric* (or, simply, *symmetric*) if for any t and any two agents i^t, $\tilde{\imath}^t \in \mathsf{D}^t(i^0)$ one has $\mathsf{x}_t(i^t) = \mathsf{x}_t(\tilde{\imath}^t) = x_t$ and $\mathsf{k}^o_{t+1}(i^t) = \mathsf{k}^o_{t+1}(\tilde{\imath}^t) = k^o_{t+1}$. A symmetric allocation is, therefore, represented by a pair of sequences $a \equiv \left\{(k^o_{t+1}, x_t) \in \mathbb{R}^3_+ \times [0, \overline{n}]\right\}_{t=0,1,2...}$ satisfying the initial condition

$$c^m_0 + b_0(n_1) + k^o_1 \leq \overline{e}_0;$$

and, for each $t \geq 1$, the feasibility constraint

$$c^o_t + n_{t+1}\left[c^m_t + b_t(n_{t+1}) + k^o_{t+1}\right] \leq F_t(k^o_t, n_t); \tag{S.1}$$

which, by writing e_t for the amount of total expenditures $e_t = c^m_t + b_t(n_{t+1}) + k^o_{t+1}$, or total income, available to each middle-aged agent at t, can be equivalently written as t

$$c^o_t + n_{t+1}e_t = F_t(k^o_t, n_t).$$

Denote by $\mathcal{S}$ the set containing all feasible, symmetric allocations.

Note that for every t, the utility obtained by the dynasty head from consumption and fertility decisions of every two alive agents i^t and $\tilde{\imath}^t$ of generation t with a symmetric allocation a satisfies $\mathcal{U}^D_t(\mathsf{a}; \tilde{\imath}^t) = U^D_t(a)$, where $U^D_t : \mathcal{S} \longrightarrow \mathbb{R}^*$ is recursively defined, for each t, by $U^D_t(a) = U^D\left(x_t, U^D_{t+1}(a)\right)$. The utility obtained by an agent of generation t with a symmetric allocation is $U_t(a) = U(x_t, U^D_{t+1}(a))$.

Elsewhere (see Conde-Ruiz *et alli*, 2010), we have proposed a notion of efficiency, referred to as Millian efficiency (or $\mathcal{M}-$efficiency), to evaluate symmetric allocations with different population size.[1] This notion results from combining the $\mathcal{A}-$dominance criterion to compare allocations with a restriction imposing symmetry on the set of allocations that can be compared using that criterion. To be more precise,[2] a feasible symmetric allocation $a \in \mathcal{S}$ is $\mathcal{M}-$efficient if there does not exist any other feasible allocation $a' \in \mathcal{S}$ such that $U_t(a') \geq U_t(a)$ for all $t \geq 0$ and $U_\tau(a') > U_\tau(a)$ for some $\tau \geq 0$. Observe that the formal definition of Millian efficiency is entirely analogous to that of symmetric, Pareto efficiency arising in OLG models with exogenous fertility. In standard, OLG economies,[3] the literature has distinguished between *static* (or *short-run*) efficiency, which means that an allocation cannot be improved upon by a reallocation of resources involving a finite number of generations, and *dynamic* (or *long run*) efficiency, which means full efficiency.

The characterization of Millian efficient allocations presented in Conde-Ruiz *et alli* (2010) –in economies without altruism– can be extended to in general setting studied in this paper. In this extended characterization Proposition S.A.1 below shows that, in a Millian efficient allocation, consumption and fertility decisions of every alive agent are completely determined by the sequence $\widehat{e} = \{\widehat{e}_t\}_{t\geq 1}$ specifying, for each t, the amount of total expenditures $\widehat{e}_t = \widehat{c}^m_t + b_t(\widehat{n}_{t+1}) + \widehat{k}^o_{t+1}$ of each agent of generation t. Formally:

[1] See also Michel and Wigniolle (2007).

[2] In our original formulation of the notion of Millian efficiency, all those symmetric allocations for which fertility rates are zero from some period t on are also ruled out from welfare comparisons.

[3] This distinction was first introduced by Balasko and Shell (1980).

Proposition S.A.1 *Every $\mathcal{M}-$efficient allocation $\widehat{a} \in \mathcal{S}$ satisfies for each $t \geq 0$*

$$\begin{aligned} U_t(\widehat{a}) &= \max_{(k^o_{t+1},x_t)\in\mathbb{R}^3_+\times[0,\overline{n}]} \Big\{ U\left(x_t, U^D_{t+1}(\widehat{a})\right) : c^m_t + b_t(n_{t+1}) + k^o_{t+1} \leq \widehat{e}_t; \\ &\quad F_{t+1}(k^o_{t+1}, n_{t+1}) - c^o_{t+1} \geq n_{t+1}\widehat{e}_{t+1} \Big\} \equiv W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{a})\right), \end{aligned} \tag{S.2}$$

*which, by Assumption A*1*, yields, for $t \geq 1$*

$$\begin{aligned} U^D_t(\widehat{a}) &= \max_{(k^o_{t+1},x_t)\in\mathbb{R}^3_+\times[0,\overline{n}]} \Big\{ U^D\left(x_t, U^D_{t+1}(\widehat{a})\right) : c^m_t + b_t(n_{t+1}) + k^o_{t+1} \leq \widehat{e}_t; \\ &\quad F_{t+1}(k^o_{t+1}, n_{t+1}) - c^o_{t+1} \geq n_{t+1}\widehat{e}_{t+1} \Big\} \equiv W^D_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{a})\right). \end{aligned}$$

Proof of Proposition S.A.1. To prove Proposition S.A.1, it suffices to show that every Millian efficient allocation must satisfy $U_t(\widehat{a}) = W_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{a}))$ for $t \geq 0$, which, by Assumption *A*1, implies that $U^D_t(\widehat{a}) = W^D_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{a}))$ must also be satisfied for each $t \geq 1$. To show that $U_t(\widehat{a}) = W_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{a}))$ is satisfied for $t \geq 0$, suppose that $\widehat{a}$ is an $\mathcal{A}-$efficient allocation, and suppose that there exists a period $\tau \geq 0$ for which the pair $(\widehat{k}^o_{\tau+1}, \widehat{x}_\tau)$ corresponding to the allocation $\widehat{a}$ is not a solution to the optimization problem in the definition of $W_\tau(\widehat{e}_\tau, \widehat{e}_{\tau+1}, U^D_{\tau+1}(\widehat{a}))$. Select now a solution $(\widetilde{k}^o_{\tau+1}, \widetilde{x}_\tau) \in \times[0, \overline{n}]$ to such optimization problem and let $\widetilde{a}$ be the allocation obtained from $\widehat{a}$ by replacing the term $(\widehat{k}^o_{\tau+1}, \widehat{x}_\tau)$ by such solution. This symmetric allocation is feasible because $(\widetilde{k}^o_{\tau+1}, \widetilde{x}_\tau)$ must satisfy $\widetilde{c}^m_\tau + b_t(\widetilde{n}_{\tau+1}) + \widetilde{k}^o_{\tau+1} \leq \widehat{e}_\tau$ and $F_{t+1}(\widetilde{k}^o_{t+1}, \widetilde{n}_{t+1}) - \widetilde{c}^o_{t+1} \geq \widetilde{n}_{t+1}\widehat{e}_{t+1}$. Also, observe that, by Assumption *A*1, the fact that $U_\tau(\widetilde{a}) > U(\widehat{x}_\tau, U^D_\tau(\widehat{a})) = U_\tau(\widehat{a})$ is satisfied, implies that $U^D_\tau(\widetilde{a}) > U^D(\widehat{x}, U^D_\tau(\widehat{a})) = U^D_\tau(\widehat{a})$ is satisfied. Therefore $U^D_t(\widetilde{a}) \geq U^D_t(\widehat{a})$ is satisfied for all $t \geq \tau$ and, since both U and U^D are monotonic, $U^D_t(\widetilde{a}) > U^D_\tau(\widehat{a})$ and $U_t(\widetilde{a}) \geq U_\tau(\widehat{a})$ are both satisfied for all $t < \tau$. That is, if the term $(\widehat{k}^o_{\tau+1}, \widehat{x}_\tau)$ is not a solution to the optimization problem in the definition of $W_\tau(\widehat{e}_\tau, \widehat{e}_{\tau+1}, U^D_{\tau+1}(\widehat{a}))$, then $\widehat{a}$ is $\mathcal{M}-$dominated by an alternative allocation $\widetilde{a}$, a contradiction that establishes that both $U_t(\widehat{a}) = W_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{a}))$ and $U^D_{t+1}(\widehat{a}) = W^D_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{\tau+1}(\widehat{a}))$ must be satisfied for each $t \geq 0$ and each $t \geq 1$, respectively. ■

Since utility and production functions are concave and differentiable, an interior solution $(\widehat{k}_{t+1}, \widehat{x}_t)$ to the optimization problem in the definitions of $W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_t(\widehat{a})\right)$ and $W^D_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_t(\widehat{a})\right)$ is characterized by the two feasibility constraints

$$\widehat{c}^m_t + b_t(\widehat{n}_{t+1}) + \widehat{k}^o_{t+1} = \widehat{e}_t, \tag{S.3}$$

and

$$F_{t+1}(\widehat{k}^o_{t+1}, \widehat{n}_{t+1}) - \widehat{c}^o_{t+1} = \widehat{n}_{t+1}\widehat{e}_{t+1}; \tag{S.4}$$

together with the first order conditions

$$\frac{D_1 U\left(\widehat{x}_t, U^D_{t+1}(\widehat{a})\right)}{D_2 U\left(\widehat{x}_t, U^D_{t+1}(\widehat{a})\right)} = \widehat{R}_{t+1}, \tag{S.5}$$

and

$$\left[b'_t(\widehat{n}_{t+1}) - \frac{D_3 U\left(\widehat{x}_t, U^D_{t+1}(\widehat{a})\right)}{D_1 U\left(\widehat{x}_t, U^D_{t+1}(\widehat{a})\right)} \right] \widehat{R}_{t+1} = \widehat{w}_{t+1} - \widehat{e}_{t+1}, \tag{S.6}$$

after writing $\widehat{R}_{t+1}$ for $\widehat{R}_{t+1} = D_1F_{t+1}(\widehat{k}^o_{t+1}, \widehat{n}_{t+1})$, and $\widehat{w}_{t+1}$ for $\widehat{w}_{t+1} = D_2F_{t+1}(\widehat{k}^o_{t+1}, \widehat{n}_{t+1})$.

Equations (S.3) to (S.5) are almost identical to those characterizing symmetric Pareto efficient allocations in an exogenous fertility setting[4] (except for the term $b_t(\widehat{n}_{t+1})$, which in that case is assumed to be zero) and are necessary for Pareto efficiency. They simply impose feasibility and that marginal rates of substitution between current and future consumption must be equal to marginal return to investments in physical capital. The Millian notion of efficiency imposes an additional condition, represented by equation (S.6) stating that if marginal willingness to pay for children is not equal to marginal costs of rearing children, then the marginal rate of return to investments in children must be equal to the rate of return to any other investment, that is,

$$\frac{\widehat{w}_{t+1} - \widehat{e}_{t+1}}{b'_t(\widehat{n}_{t+1}) - \frac{D_3U\left(\widehat{x}_t, U^D_{t+1}(\widehat{a})\right)}{D_1U\left(\widehat{x}_t, U^D_{t+1}(\widehat{a})\right)}} = \widehat{R}_{t+1}.$$

That is, in a Millian efficient allocation $\widehat{a}$, consumption and fertility decisions of the agents are completely determined by the total amount of resources $\widehat{e}_t$ available to the agent, the total amount of resources $\widehat{e}_{t+1}$ available to each descendant and the average utility $U^D_{t+1}(\widehat{a})$ that the agent obtains from consumption decisions of her descendants. Given a sequence $e = \{e_t : e_0 = \overline{e}_0\}_{t\geq 0}$, For each $t \geq 0$, write e^t for the finite sequence $e^t = (e_0, e_1, e_2, ..., e_t)$; and, write e^{-t} for the infinite sequence of non-negative real numbers $e^{-t} = \{e_\tau\}_{t\geq t+1} = (e_{t+1}, e^{-(t+1)})$. A consequence of Proposition S.A.1 is that, for every $t \geq 0$, the utility that any agent of generation t obtains with a Millian efficient allocation $\widehat{a}$ is entirely determined by the sequence $(\widehat{e}_t, \widehat{e}^{-t})$. Then, it follows from (S.2) that the utility obtained by any agent of generation t in a Millian efficient allocation $\widehat{a}$ can be written as

$$U_t(\widehat{a}) = w_t(\widehat{e}_t, \widehat{e}^{-t}) \equiv W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, w^D_{t+1}(\widehat{e}_{t+1}, \widehat{e}^{-(t+1)})\right),$$

where $w^D_t(\widehat{e}_t, \widehat{e}^{-t})$ is recursively defined, for any $(\widehat{e}_t, \widehat{e}^{-t})$, by

$$\begin{aligned} w^D_t(\widehat{e}_t, \widehat{e}^{-t}) &= W^D_t(\widehat{e}_t, \widehat{e}_{t+1}, w^D_{t+1}(\widehat{e}_{t+1}, \widehat{e}^{-(t+1)})) = \\ &= W^D_t(\widehat{e}_t, \widehat{e}_{t+1}, W^D_{t+1}(\widehat{e}_{t+1}, \widehat{e}_{t+2}, w^D_{t+2}(\widehat{e}_{t+2}, \widehat{e}^{-(t+2)})) = ... \end{aligned}$$

In classical, OLG economies[5] with endogenous fertility, the literature has distinguished between *static* (or *short-run*) Pareto efficiency, which means that an allocation cannot be improved upon by a reallocation of resources involving a finite number of generations, and *dynamic* (or *long run*) efficiency, which means full efficiency. This distinction is also applicable to the notion of Millian efficiency, and we shall distinguish between static and dynamic Millian efficient allocations. Formally:

Definition 1 *A symmetric, feasible allocation $\widehat{a} \in \mathcal{S}$ is statically $\mathcal{M}-$efficient if there does not exist another symmetric, feasible allocation $\widetilde{a} \in \mathcal{S}$ and a finite period $T \geq 0$ such that*

i) $(\widehat{k}^o_{t+1}, \widehat{x}_t) = (\widetilde{k}^o_{t+1}, \widetilde{x}_t)$ *for all* $t > T$;

[4] See e.g., Blanchard and Fisher (1989, p.99).

[5] This distinction was introduced first by Balasko and Shell (1980).

ii) for all t such that $0 \leq t \leq T$ one has $U_t(\widetilde{a}) \geq U_t(\widehat{a})$; and,

iii) there exists t such that $0 \leq t \leq T$ and $U_t(\widetilde{a}) > U_t(\widehat{a})$ is satisfied.

As we show in Conde-Ruiz *et alli* (2010), condition (S.2) in Proposition S.A.1 is sufficient to characterize every statically Millian efficient allocation in economies without altruism. In our extension to environments with altruism, static, Millian efficiency can be characterized as follows:

Proposition S.A.2 *i) Every $\mathcal{M}-$efficient allocation $\widehat{a} \in \mathcal{S}$ satisfies (S.2) and*

$$\begin{aligned} U_t^D(\widehat{a}) &= \max\left\{w_t^D(\widehat{e}_t, e^{-t}) : e^{-t} \geq \widehat{e}^{-t}\right\} \equiv v_t^D(\widehat{e}_t, \widehat{e}^{-t}) \\ &= \max\left\{W_t^D(\widehat{e}_t, e_{t+1}, v_t^D(e_{t+1}, \widehat{e}^{-(t+1)}) : e_{t+1} \geq \widehat{e}_{t+1})\right\}, \end{aligned} \tag{S.7}$$

for each $t \geq 1$; and

$$\begin{aligned} U_0(\widehat{a}) &= v_0(\overline{e}_0, \widehat{e}^{-0}) \equiv \max_{a \in \mathcal{S}}\left\{U_0(x) : e^{-0} \geq \widehat{e}^{-0}\right\} \\ &= \max\left\{W_0(\overline{e}_0, e_1, v_1^D(e_1, \widehat{e}^{-1})) : e_1 \geq \widehat{e}_1\right\}, \text{ for } t = 0. \end{aligned} \tag{S.8}$$

ii) Moreover, an allocation $\widehat{a}$ satisfying the necessary conditions in i) is statically $\mathcal{M}-$efficient.

Proof of Proposition S.A.2. To prove that $U_t^D(\widehat{a}) = v_t^D(\widehat{e}_t, \widehat{e}^{-t})$ must be satisfied for each $t \geq 1$, write $v_t^D(\widehat{e}_t, \widehat{e}^{-t})$ as

$$v_t^D(\widehat{e}_t, \widehat{e}^{-t}) = \max_{e_{t+1} \geq 0}\left\{W_t^D\left(\widehat{e}_t, e_{t+1}, v_{t+1}^D(e_{t+1}, \widehat{e}^{-(t+1)})\right)\right\},$$

and suppose that there exists a period $\tau \geq 0$ for which the sequence $\widehat{e}^{-\tau}$ corresponding to the allocation $\widehat{a}$ is not a solution to the optimization problem in the definition of $v_\tau^D(\widehat{e}_\tau, \widehat{e}^{-\tau})$. Select now a sequence $\widetilde{e}^{-\tau} = \{\widetilde{e}_s\}_{s>\tau}$ such that $v_\tau^D(\widehat{e}_\tau, \widetilde{e}^{-\tau}) > v_\tau^D(\widehat{e}_\tau, \widehat{e}^{-\tau})$ is satisfied for $\tau \geq 0$, and let now $\widetilde{a}$ be the symmetric allocation for which $U_t^D(\widetilde{a}) = v_t^D(\widehat{e}_t, \widetilde{e}^{-t})$ is satisfied. Note that

$$U_t^D(\widetilde{a}) = W_t^D\left(\widehat{e}_t, \widetilde{e}_{t+1}, U_{t+1}^D(\widetilde{a})\right) \geq W_t^D\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{a})\right) = U_t^D(\widehat{a})$$

must be satisfied for each $t \geq 1$. Moreover, the above inequality must be satisfied as a strict inequality for $t = \tau$. Finally, Assumption $A3$ and the fact that $U_t^D(\widetilde{a}) \geq U_t^D(\widehat{a})$ is satisfied implies that

$$U_t(\widetilde{a}) = W_t\left(\widetilde{e}_t, \widetilde{e}_{t+1}, U_{t+1}^D(\widetilde{a})\right) \geq W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{a})\right) = U_t(\widehat{a})$$

is also satisfied for each $t \geq 1$, and with strict inequality for $t = \tau$. This implies that the allocation $\widehat{a}$ is not $\mathcal{M}-$efficient, a contradiction that establishes that $U_t^D(\widehat{a}) = v_t^D(\widehat{e}_t, \widehat{e}^{-t})$ must be satisfied. Taking this into account, $U_0(\widehat{a}) = \max_{e_1 \geq 0}\left\{W_0(\overline{e}_0, e_1, v_1^D(e_1, \widehat{e}^{-1}))\right\}$ follows straightforwardly, which completes the proof of the $i)$ statement in Proposition S.A.2.

To prove $ii)$, let $\widehat{a} \in S$ be an allocation satisfying the necessary conditions in $i)$. To prove that $\widehat{a}$ is statically $\mathcal{M}-$efficient, we proceed by showing that if there exists an allocation $\widetilde{a}$ that $\mathcal{M}-$dominates the allocation $\widehat{a}$, then there must exist an infinite subsequence $\mathcal{T} = \{t_1, t_2, t_3, ...\}$ such that $\widetilde{e}_\tau < \widehat{e}$ for each $t \in \mathcal{T} \geq 1$. To prove this statement, observe that the fact that

$U_0(\widehat{a}) = v_0(\overline{e}_0, \widehat{e}^{-0})$ and the fact that v_0 is non-increasing in e^{-0} is satisfied imply that $\widetilde{e}_{t_1} < \widehat{e}_{t_1}$ must be satisfied for some period $t_1 \geq 0$. Since $v^D_{t_1}$ is strictly increasing in e_{t_1} and non-increasing in e^{-t_1}, the fact that $\widetilde{e}^{-t_1} \geq \widehat{e}^{-t_1}$ is satisfied implies that

$$U^D_{t_1}(\widetilde{a}) < v^D_{t_1}(\widehat{e}_{t_1}, \widehat{e}^{-t_1}) = W^D_{t_1}\left(\widehat{e}_{t_1}, \widehat{e}_{t_1+1}, v_{t_1}(\widehat{e}_{t_1+1}, \widehat{e}^{-(T_1+1)})\right) = U^D_{t_1}(\widehat{a})$$

must be satisfied, which, in turn, yields

$$U_0(\widetilde{a}) = w_0(\overline{e}_0, \widetilde{e}_1, \widetilde{e}_2, ..., \widetilde{e}_{t_1-1}, v^D_{t_1}(\widehat{e}_{t_1}, \widehat{e}^{-t_1})) < U_0(\widehat{a}),$$

which contradicts the assumption imposing that $\widetilde{a}$ $\mathcal{M}-$dominates the allocation $\widehat{a}$. Therefore, $U_0(\widetilde{a}) \geq U_0(\widehat{a})$ can be satisfied only if there exists t_2 for which $\widetilde{e}_{t_2} < \widehat{e}_{t_2}$ and $W^D_{t_2}(\widetilde{e}_{t_2}, \widetilde{e}_{t_2+1}, U^D_{t_2+1}(\widetilde{a})) > U_{t_2}(\widehat{a})$ is satisfied. By applying the argument recursively, the existence of the subsequence $\mathcal{T}$ is established. Also, since the allocation $\widehat{a}$ can only be dominated by a reallocation of resources involving a infinite sequence of periods of time, the allocation $\widehat{a}$ must be statically $\mathcal{M}-$efficient, which completes the proof of Proposition S.A.2. ■

A particular Millian efficient allocation is the allocation a^* for which the restriction $e^{-t} \geq \widehat{e}^{-t}$ is not binding, that is, the allocation that maximizes the utility of the dynasty head among feasible, symmetric allocations. In such allocation,

$$U_0(x^*) = V_0(\overline{e}_0) =: \max_{e_1 \in \mathbb{R}_+} W_0\left(\overline{e}_0, e_1, V^D_1(e_1)\right)$$

where, for each $t \geq 0$, $V^D_t : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is defined, for every $e_t \in \mathbb{R}_+$, by

$$V^D_t(e_t) = v^D_t(\widehat{e}_t, 0) = \max_{e_{t+1} \in \mathbb{R}_+} W_t\left(e_t, e_{t+1}, v^D_{t+1}(e_{t+1}, 0)\right).$$

We should point out that, in the characterization given above, the fact that fertility is endogenous does not play any specific role. Therefore, conditions characterizing static efficiency in an exogenous fertility setting are almost identical to (S.7) and (S.8). The only difference between the characterization of statically efficient paths for these two notions of efficiency is that, in an exogenous fertility problem, is not a choice variable in the optimization problem in the definition of $W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_t(\widehat{a})\right)$ and $W^D_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{a})\right)$. The necessary conditions in Proposition S.A.2.*i*) imply that, if the dynasty head (or, in settings with finite horizon altruism, any agent) is restricted to choose a symmetric allocation such that $e \geq \widehat{e}$, she cannot do better than she does with $\widehat{a}$. These necessary conditions Conditions (S.7) and (S.8) imply the static efficiency of an allocation $\widehat{a}$ because, for any allocation a that $\mathcal{M}-$dominates $\widehat{a}$, one must have $e_{t_1} < \widehat{e}_{t_1}$ for some $t_1 > 0$, which, in turn, implies that $e_{t_2} < \widehat{e}_{t_2}$ for some $t_2 > t_1$, and so forth.

As for the notion of Pareto efficiency with exogenous fertility, the static, Millian efficiency of a given allocation $\widehat{a}$ does not preclude that all living agents can improve their welfare by reducing total resources accumulated by some generations of agents with the allocation $\widehat{a}$. In the proof of Proposition S.A.3 below, we show that if a statically efficient allocation $\widehat{a}$ is not fully $\mathcal{M}-$efficient (or, as it is usually found in the literature, dynamically $\mathcal{M}-$efficient) there must exist an infinite sequence $\{e_t\}_{t\geq 0}$ satisfying

$$e_t < \widehat{e}_t \text{ and } W_t(e_t, e_{t+1}, U^D_t(\widehat{x})) \geq W_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_t(\widehat{a})), \text{ for each } t \geq 1, \tag{S.9}$$

which, by Assumption $A1$ implies in turn that the sequence $\{e_t\}_{t\geq 0}$ also satisfies

$$e_t < \widehat{e}_t \text{ and } W_t^D(e_t, e_{t+1}, U_t^D(\widehat{x})) \geq W_t^D(\widehat{e}_t, \widehat{e}_{t+1}, U_t^D(\widehat{a})), \text{ for each } t \geq 1.$$

Thus, a sufficient condition ensuring dynamic efficiency of a statically Millian efficient path $\widehat{a}$ is that such sequence $\{e_t\}_{t\geq 0}$ does not exist.

Here, an important difference between the properties of Millian efficient allocations and Pareto efficient allocations –with exogenous fertility– arise. In the latter case, the set of feasible allocations is convex, and the concavity of utility function U implies that each indirect utility functions $W_t(\cdot, U_t^D(\widehat{x}))$ must be quasiconcave. It is straightforward to show that, when $U_t(\widehat{x}) = W_t(\widehat{e}_t, \widehat{e}_{t+1}, U_t^D(\widehat{x}))$ is satisfied and $W_t(\cdot, U_t^D(\widehat{x}))$ is quasiconcave, condition (S.9) can be written as

$$e_t < \widehat{e}_t \text{ and } \widehat{e}_t - e_t \geq -\frac{D_1 W_t\left[\widehat{e}_t, \widehat{e}_{t+1}, U_t^D(\widehat{a})\right]}{D_2 W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_t^D(\widehat{a})\right)}\left(\widehat{e}_{t+1} - e_{t+1}\right) = \frac{\widehat{R}_{t+1}}{\widehat{n}_{t+1}}\left[\widehat{e}_{t+1} - e_{t+1}\right]$$

for each $t \geq 1$. Proceeding recursively, it is straightforward to show[6] that a sequence $\{e_t\}_{t\geq 0}$ satisfying (S.9) cannot exist –which implies that $\widehat{a}$ is dynamically efficient– if

$$\lim_{T\to\infty} \inf \left(\prod_{t=0}^{T} \frac{\widehat{e}_{t+1}\widehat{n}_{t+1}}{\widehat{e}_t \widehat{R}_{t+1}}\right) = 0 \tag{S.10}$$

is satisfied. In particular, for an allocation $\widehat{a}$ such that $\lim_{t\to\infty} \frac{\widehat{e}_{t+1}}{\widehat{e}_t} = 1$, $\lim_{t\to\infty} \widehat{n}_{t+1} = n^*$ and $\lim_{t\to\infty} \widehat{R}_{t+1} = R^*$, condition (S.10) can be written as the standard dynamic efficiency condition

$$R^* > n^*.$$

It is in the analysis of dynamic efficiency where an important difference between the properties of Millian efficient allocations and those of Pareto efficient allocations –with exogenous fertility– arise. When fertility decisions are endogenous, the indirect utility function $W_t\left(\cdot, U_t^D(\widehat{a})\right)$ is not, in general, quasiconcave. Due to these non-convexities, standard dynamic efficiency conditions need not be valid to identify efficient paths. Yet, a sufficient dynamic efficiency condition can be obtained as follows.

For a given $\left(\widehat{e}_t, \widehat{e}_{t+1}, \widehat{u}_{t+1}^D\right)$, define

$$\pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, \widehat{u}_{t+1}^D\right) = \inf_{(e_t, e_{t+1}) << (\widehat{e}_t, \widehat{e}_{t+1})} \left\{ \frac{\widehat{e}_{t+1} - e_{t+1}}{\widehat{e}_t - e_t} : W_t\left(e_t, e_{t+1}, \widehat{u}_{t+1}^D\right) \geq W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, \widehat{u}_{t+1}^D\right) \right\}.$$

Notice that, when $W_t\left(\cdot, \widehat{u}_{t+1}^D\right)$ is quasiconcave, the number $\pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, \widehat{u}_{t+1}^D\right)$ corresponds to the slope of a indifference curve defined by $W_t\left(e_t, e_{t+1}, \widehat{u}_{t+1}^D\right) = W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, \widehat{u}_{t+1}^D\right)$ evaluated at $(\widehat{e}_t, \widehat{e}_{t+1})$. That is, for quasiconcave indirect utility functions we have

$$\pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{a})) = -\frac{D_1 W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{a})\right)}{D_2 W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{a})\right)} = \frac{\widehat{R}_{t+1}}{\widehat{n}_t}.$$

[6]See, e.g., Lemma 5.4 in Balasko and Shell (1980).

However, when indirect utility functions are not quasiconcave, the number $\pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{a}))$ corresponds to the steepest slope (on the set $(e_t, e_{t+1}) << (\widehat{e}_t, \widehat{e}_{t+1})$) of the indifference curve defined by $W_t\left(e_t, e_{t+1}, U^D_t(\widehat{a})\right) = W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_t(\widehat{a})\right)$. Therefore

$$\pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{a})) \leq -\frac{D_1 W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_t(\widehat{a})\right)}{D_2 W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_t(\widehat{a})\right)} = \frac{\widehat{R}_{t+1}}{\widehat{n}_{t+1}},$$

and condition (S.9) can be written as

$$e_t < \widehat{e}_t \text{ and } \widehat{e}_t - e_t \geq \pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{a}))\left(\widehat{e}_{t+1} - e_{t+1}\right) \text{ for each } t \geq 1.$$

Using this notation, in Proposition S.A.3 below we shown that the sufficient condition for dynamic efficiency in Conde-Ruiz *et alli* (2010, Prop.5) applies also to this general setting and provide a sufficient condition for dynamic efficiency that uses the sequence of implicit prices $\{(\widehat{R}_{t+1}, \widehat{w}_{t+1})\}_{t\geq 0}$ associated to a statically $\mathcal{M}-$efficient allocation $\widehat{a}$ satisfying (S.10).

Proposition S.A.3 *Consider a statically Millian efficient allocation $\widehat{a} \in S$ satisfying (S.10). If*

$$\lim_{T\to\infty} \inf \left(\frac{\widehat{e}_T}{\prod_{t=0}^{T} \pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{a})\right)} \right) = 0 \tag{S.11}$$

is satisfied, then $\widehat{a}$ is also (dynamically) efficient. Furthermore, a sufficient condition ensuring (S.11) is that

$$\lim_{T\to\infty} \left(b'_T(0) - \frac{\widehat{w}_{T+1}}{\widehat{R}_{T+1}} \right) \frac{\widehat{n}_{T+1}}{\widehat{e}_T} > 0,$$

is satisfied.

Proof of Proposition S.A.3. To prove Proposition S.A.3, we first show that, if there exists an allocation $\widetilde{a}$ that $\mathcal{M}-$dominates the allocation $\widehat{a}$, then there must exist an allocation a that also dominates $\widehat{a}$ and satisfies, for $t \geq 1$

$$U_t(a) = U_t(\widehat{a}),\ U^D_t(a) = U^D_t(\widehat{a}), \text{ and } e_t < \widehat{e}_t.$$

To prove this statement, assume, without loss of generality, that $\widetilde{a}$ satisfies the necessary conditions for Millian efficiency in Proposition S.A.2. Taking this into account, an allocation a satisfying the required properties can be constructed from the allocation $\widetilde{a}$ as follows. Pick up any period $\tau \geqslant 1$ for which $U_\tau(\widetilde{a}) = W_\tau(\widetilde{e}_\tau, \widetilde{e}_{\tau+1}, U^D_{\tau+1}(\widetilde{a})) > U_\tau(\widehat{a})$ is satisfied and select $e^1_\tau < \widetilde{e}^1_\tau$ in such a way that $W_\tau(e^1_\tau, \widetilde{e}_{\tau+1}, U^D_{\tau+1}(\widetilde{a})) = U_\tau(\widehat{a})$ is satisfied. Notice that, by Assumption $A1$ one must have $W^D_\tau(e^1_\tau, \widetilde{e}_{\tau+1}, U^D_{\tau+1}(\widetilde{a})) = U^D_\tau(\widehat{a})$. Then let a^1 be the allocation obtained from $\widetilde{a}$ by replacing the term $(\widetilde{k}^o_\tau \widetilde{x}_{,\tau})$ by the solution $(k^{o1}_\tau,\ x^1_\tau)$ to the optimization problem in the definition of $W_\tau(e^1_\tau, \widetilde{e}_{\tau+1}, U^D_{\tau+1}(\widetilde{a}))$. Note that, since $\widetilde{a}$ satisfies the necessary conditions in Proposition S.A.2 and $w^D_{\tau-1}$ is non-increasing in e_t, we have $U^D_{\tau-1}(a^1) = W^D_{\tau-1}(\widetilde{e}_{\tau-1}, e^1_\tau, W^D_\tau(e^1_\tau, \widetilde{e}_{\tau+1}, U^D_{\tau+1}(\widetilde{a}))) \geq U^D_\tau(\widetilde{a})$. Thus, $U^D_{\tau-1}(x^1) \geq U^D_{\tau-1}(\widehat{a})$ and, hence, $U_{\tau-1}(a^1) \geq U_{\tau-1}(\widehat{a})$ must be satisfied, that is, the allocation a^1 dominates the allocation $\widetilde{a}$. Proceeding iteratively, it is straightforward to construct an allocation $\overline{a}$ satisfying the required properties for each t: $1 \leq t \leq \tau$, which taking into account that τ has been selected arbitrarily among those periods for which $U_t(\overline{x}) > U_t(\widehat{x})$ is satisfied, establishes Step 1). The remaining of the proof is exactly analogous to the proof of Propositions and in Propositions 4 and 5 in Conde-Ruiz *et alli* (2010). ■

S.A.1 Dynamic efficiency and transversality conditions in dynastic problems.

A particular Millian efficient allocation is the one maximizing the utility of the dynasty head among symmetric allocations, that is, the Millian efficient allocation $\widehat{a}$ for which

$$U_t^D(\widehat{a}) = v_t^D(\widehat{e}_t, 0) \equiv V_t^D(\widehat{e}_t), \text{ for } t \geq 1$$

and

$$U_0(\widehat{a}) = \max_{e_1 \geq 0} \left\{ W_0(\overline{e}_0, e_1, V_1^D(e_1)) \right\}.$$

Note that such allocation must satisfy the necessary condition for dynastic maximization

$$U_t^D(\widehat{a}) = \max_{e_{t+1} \geq 0} \left\{ W_t^D(\widehat{e}_t, e_{t+1}, w_{t+1}^D(e_{t+1}, \widehat{e}^{-(t+1)}) \right\}, \text{ for each } t \geq 1. \tag{S.12}$$

However, there exist many allocations satisfying this condition. In concave settings, the sequence $\{\widehat{e}_t\}_{t\geq 0}$ corresponding to a dynastic optimum can be identified as that satisfying the transversality condition

$$\lim_{T\to\infty} \inf \prod_{t=0}^{T} \left(\frac{\widehat{e}_{t+1}\widehat{n}_{t+1}}{\widehat{e}_t \widehat{R}_{t+1}} \right) = 0. \tag{S.13}$$

Unfortunately, standard transversality conditions might not work in non-convex settings. Yet, it is straightforward to show that the dynamic efficiency conditions in Proposition S.A.3 are sufficient transversality conditions ensuring that an allocation satisfying (S.12) corresponds to the dynastic optimum.

S.A.2 Other environments with finite horizon altruism

Some of the results obtained above can be easily extended to other environments in which the agents care about some of the decisions of their immediate descendants, even though Assumptions $A1$ and $A2$ are not satisfied. As an example, suppose the agents care about their immediate descendants' wealth during their second period of life, that is, preferences are represented by utility functions of the form

$$\mathcal{U}_t(\mathsf{a}, i^t) = U\left(\mathsf{x}(i^t), \frac{1}{\mathsf{n}_{t+1}(i^t)} \int_{\mathsf{D}_{t+1}(i^t)} u^D(\mathsf{e}_{t+1}(i^t, i_{t+1}))di_{t+1} \right);$$

where U is strictly increasing and concave on $\mathbb{R}^4_+$, and u^D is also strictly increasing and concave. Note that, when applied to symmetric allocations, such preferences give rise to a utility function U_t defined, for each $t \geq 0$ and each symmetric allocation a, by

$$U_t(a) = U\left(x_t, u^D(e_{t+1})\right).$$

With these preferences, Assumption $A2$ is not satisfied. Yet, in this case, it is easy to prove that $U_t(\widehat{a}) = W_t(\widehat{e}_t, \widehat{e}_{t+1}, u_t^D(\widehat{e}_{t+1}))$ must hold for any Millian efficient allocation. Differently from the characterization given in Proposition S.A.3, statically efficient, Millian efficient allocations can be characterized, in this case, as those satisfying, for each $t \geq 0$

$$U_t(\widehat{a}) = \max\left\{ W_t(\widehat{e}_t, e_{t+1}, u_t^D(e_{t+1})) : e_{t+1} \geq \widehat{e}_{t+1} \right\}.$$

Dynamic efficiency conditions, however, are entirely analogous to those in Proposition S.A.3.

In the setting described, the main results obtained in the paper must be slightly modified. First, if the function u^D is strictly concave, then every $\mathcal{A}-$efficient allocation must be symmetric and can be characterized as a Millian efficient allocation for which, at each point in time, $\widehat{e}_{t+1}$ solves, without constraints, the optimization problem

$$\max\left\{W_t(\widehat{e}_t, e_{t+1}, u_t^D(e_{t+1})) : e_{t+1} \geq 0\right\}.$$

Thus, although $\mathcal{A}-$efficiency does not drive the economy to a collapse, the set of $\mathcal{A}-$efficient allocations does reduce, typically, to a singleton. Second of all, when the functions determining the welfare attributed to the unborn satisfy Property $\boldsymbol{S}$ (see Section 5 in our paper), every symmetric, $\mathcal{P}-$efficient allocation can be characterized as a Millian efficient allocation. Third, competitive equilibria –which in this setting are necessarily symmetric– are statically efficient and, under certain conditions, dynamically efficient and $\mathcal{P}-$efficient. A competitive equilibrium might also be $\mathcal{A}-$efficient, but only if the non-negativity constraint on gifts and bequests is not binding.

As another example, suppose that the agents care about their immediate descendants' middle-aged consumption during their second period of life, that is, preferences are represented by utility functions of the form

$$\mathcal{U}_t(\mathsf{a}; i^t) = U\left(\mathsf{x}(i^t), \frac{1}{\mathsf{n}_{t+1}(i^t)}\int_{\mathsf{D}_{t+1}(i^t)} u^D(\mathsf{c}^m_{t+1}(i^t, i_{t+1}))di_{t+1}\right);$$

where U is strictly increasing and concave on $\mathbb{R}^4_+$ and u^D is also strictly increasing and concave. In this case, when applied to symmetric allocations, such preferences give rise to a utility function U_t defined, for each $t \geq 0$ and each symmetric allocation a, by

$$U_t(a) = U\left(x_t, u^D(\widehat{c}^m_{t+1})\right).$$

it is straightforward to show that, in any $\mathcal{M}-$efficient allocation $\widehat{a}$ and for every t, the vector $a_t = (c^m_t, c^o_{t+1}, n_{t+1}, k^o_{t+1})$ must solve a problem closely analogous to that in the definition of $W_t(\widehat{e}_t, \widehat{e}_{t+1}, u^D(\widehat{c}^m_{t+1}))$, with an additional constraint of the form $c^m_{t+1} \geq \widehat{c}^m_{t+1}$. Thus, those allocations for which $U_t(\widehat{a}_t) = W_t(\widehat{e}_t, \widehat{e}_{t+1}, u^D(\widehat{c}^m_{t+1}))$–that is, those allocations for which the constraint $c^m_t \geq \widehat{c}^m_t$ is not binding– trivially satisfy such necessary condition of $\mathcal{M}-$efficiency and, therefore, form a subclass of all possible $\mathcal{M}-$efficient allocations. By writing $C_t(\widehat{e}_t, \widehat{e}_{t+1}, u^D(\widehat{c}^m_{t+1}))$ for the solution to the optimization problem in the definition of $W_t(\widehat{e}_t, \widehat{e}_{t+1}, u^D(\widehat{c}^m_{t+1}))$ and by letting $w^D_t(e_t, e^{-t})$ be recursively defined by $w^D_t(\widehat{e}_t, \widehat{e}^{-t}) = u^D(C_t(\widehat{e}_t, \widehat{e}_{t+1}, w^D_{t+1}(\widehat{e}_{t+2}, \widehat{e}^{-(t+2)}))$, it is straightforward to show condition $i)$ in Proposition S.A.2 is necessary and sufficient to characterize statically $\mathcal{M}-$efficient paths within the class of allocations for which $U_t(\widehat{x}_t) = W_t(\widehat{e}_t, \widehat{e}_{t+1}, u^D_t(\widehat{c}^m_{t+1}))$ is satisfied. As for the previous example, dynamic efficiency conditions, however, are entirely analogous to those in Proposition S.A.3.

In such a setting, no matter whether the constraints $c^m_t \geq \widehat{c}^m_t$ are binding or not, the qualitative results obtained in this paper extend to environments with this type of altruism. In particular, $\mathcal{A}-$efficiency –when applied with the *Birth-Date criterion*– reduces to dynastic maximization, which implies that $\widehat{e}_2 = 0$ and $\widehat{c}^m_{t+2} = 0$ must be satisfied and, therefore, drives the economy to a collapse. Also, similar results to those in Theorems 1 and 3 in the main paper hold.

Appendix S.B: On concavity of value functions in a model with infinite horizon altruism

In this Supplementary Appendix S.B, we provide conditions ensuring the concavity of value functions, as well as an example of an economy for which value functions are non-concave.

Only a few papers have explicitly studied whether or not value functions arising in standard models of endogenous fertility are concave. Two important exceptions are Álvarez (1999) and Qi and Kadaya (2010), although they both focus on value functions arising when the dynasty head is restricted to select symmetric allocations. The first paper focuses on Barro and Becker's (1989) model and shows that the optimization problem in the definition of the value function can be transformed in such a way that the feasible set is convex and the utility function is concave, which suffices to ensure concavity of the value function. The latter provides conditions ensuring the concavity of symmetric value functions in a more general setting, but it does it at a cost: fertility choices must be bounded from below by a strictly positive number.

S.B.1 Concavity of value functions in an extension of Barro and Becker's model

Although Alvarez's paper focuses on the restricted value functions arising in Barro and Becker's model, its arguments can be easily extended to the unrestricted value functions arising in more general models with truly overlapping generations. That is, to models for which costs of rearing children are linear –i.e., $b_t(n_{t+1}) = bn_{t+1}$– and that utility functions adopt the form

$$U^D\left(x_t, u^D_{t+1}\right) = u(x_t) + \beta n^{\gamma}_{t+1} u^D_{t+1} = u(x_t) + \beta n^{\gamma}_{t+1} U^D\left(x_{t+1}, u^D_{t+2}\right) = ...;$$

where $u(\cdot)$ is concave and, since we are imposing that U^D must be non decreasing, $0 \leq \gamma < 1$ is satisfied whenever $u(\cdot)$ is positive valued and $\gamma \leq 0$ is satisfied whenever $u(\cdot)$ is negative valued.

Assume, without loss of generality, that $\mathcal{D} \equiv \mathcal{D}^0$ holds, so that fertility choices are completely detemined by a sequence $\mathsf{n} = \left\{\mathsf{n}_{t+1} : [0,\overline{n}]^t \to [0,\overline{n}]\right\}_{t=0,1,2}$. In this context, the utility $\mathcal{U}^D_t(a; i^t)$ that the dynasty head obtains from consumption and fertility decisions of her descendants of the dynasty initiated by agent i^t at t can be written as

$$\mathcal{U}^D_t(\mathsf{a}; i^t) = \sum_{\tau=t}^{\infty} \beta^{\tau-t} \int_{\mathsf{D}^{\tau}(i^t)} \left(\mathsf{N}_{\tau}(i^{\tau})\right)^{\gamma} u(\mathsf{x}_{\tau}(i^{\tau})) di^{\tau},$$

where $\mathsf{N}_t(i^t) = 1$; $\int_{\mathsf{D}^t(i^t)} \left(\mathsf{N}_t(i^t)\right)^{\gamma} u(\mathsf{x}_t(i^t)) di^t = u(\mathsf{x}_t(i^t))$ and, for each $\tau > t$ and each $i^{\tau} \in \mathsf{D}^{\tau}(i^t)$, $\mathsf{N}_{\tau}(i^{\tau}) = \mathsf{N}_t(i^{\tau-1}, i_{\tau})$ is recursively defined by

$$\begin{aligned} \mathsf{N}_t(i^{\tau}) &= \mathsf{N}_{\tau-1}(i^{\tau-1})\mathsf{n}_{\tau}(i^{\tau-1}) = \\ &= \mathsf{N}_{\tau-2}(i^{\tau-2})\mathsf{n}_{\tau-2}(i^{\tau-2})\mathsf{n}_{\tau-1}(i^{\tau-1}) == \prod_{s=t+1}^{\tau} \mathsf{n}_s(i^{s-1}). \end{aligned}$$

Recall also that $\mathcal{V}^D_1(e_t)$ can be defined as the maximum of $\mathcal{U}^D_t(a; i^t)$ (for an arbitrary i^t) among all sequences $\left\{(\mathsf{k}^o_{\tau+1}, \mathsf{x}_{\tau}) : [0,\overline{n}]^{\tau} \longrightarrow \mathbb{R}^3_+ \times [0,\overline{n}]\right\}_{\tau \geq t}$ satisfying the feasibility constraints

$$\mathsf{c}^m_t(i^t) + b\mathsf{n}_{t+1}(i^t) + \mathsf{k}^o_{t+1}(i^t) = e_t;$$

for $\tau = t$; and

$$\int_{\mathsf{D}^\tau(i^t)} \left(\mathsf{c}_t^o(i^{\tau-1}) + \mathsf{n}_\tau(i^{\tau-1}) \left[\mathsf{c}_\tau^m(i^\tau) + b\mathsf{n}_{\tau+1}(i^\tau) + \mathsf{k}_{\tau+1}^o(i^\tau)\right]\right) di^\tau \leq \int_{\mathsf{D}^\tau(i^t)} F_\tau(\mathsf{k}_\tau^o(i^{\tau-1}), \mathsf{n}_\tau(i^{\tau-1}))di^\tau,$$

for $\tau > t$. Write now, for each $\tau > t$ and each $i^\tau \in \mathsf{D}^\tau(i^t)$, $\mathsf{X}_\tau(i^\tau)$ for $\mathsf{X}_\tau(i^\tau) = \mathsf{N}_\tau(i^\tau)\mathsf{x}_\tau(i^\tau)$ and $\mathsf{K}_{\tau+1}(i^\tau)$ for $\mathsf{K}_{\tau+1}(i^\tau) = \mathsf{N}_\tau(i^\tau)\mathsf{k}_{\tau+1}^o(i^\tau)$. With this notation, the optimization problem in the definition of $\mathcal{V}_1^D(e_t)$ is equivalent to that of choosing $\left\{(\mathsf{K}_{\tau+1}^o, \mathsf{X}_\tau) : \mathbb{R}_+^\tau \longrightarrow \mathbb{R}_+^4\right\}_{\tau \geq t}$ to maximize

$$\begin{aligned}\mathcal{U}_t^D(\mathsf{x}; i^t) &= \sum_{\tau=t}^{\infty} \beta^{\tau-t} \int_{\mathsf{D}^\tau(i^t)} \mathsf{N}_\tau(i^\tau)^\gamma u\left(\frac{\mathsf{X}_\tau(i^\tau)}{\mathsf{N}_\tau(i^\tau)}\right) di^\tau \\ &= \sum_{\tau=t}^{\infty} \beta^{\tau-t} \int_{\mathsf{D}^\tau(i^t)} U^{Al}(\mathsf{N}_\tau(i^\tau), \mathsf{X}(i^\tau))di^\tau,\end{aligned}$$

among all sequences $\left\{(\mathsf{K}_{\tau+1}^o, \mathsf{X}_\tau) : \mathbb{R}_+^\tau \longrightarrow \mathbb{R}_+^4\right\}_{\tau \geq t}$ satisfying

$$\mathsf{C}_t^m(i^t) + b\mathsf{N}_{t+1}(i^t) + \mathsf{K}_{t+1}(i^t) = e_t;$$

for $\tau = t$; and

$$\int_{\mathbb{R}_+^\tau} \left(\mathsf{C}_t^o(i^{\tau-1}) + \mathsf{C}_\tau^m(i^\tau) + b\mathsf{N}_\tau(i^\tau) + \mathsf{K}_{\tau+1}(i^\tau)\right) di^\tau \leq \int_{\mathbb{R}_+^\tau} F_\tau(\mathsf{K}_\tau(i^{\tau-1}), \mathsf{N}_\tau(i^{\tau-1}))di^\tau.$$

for $\tau > t$. Therefore, when U^{Al} is concave, the optimization problem in the definition of $\mathcal{V}_t^D(e_t)$ is a concave program with a concave objective function and a sequence of constraints that define a convex set. Thus, if $\mathcal{V}_t^D(e_t)$ is well defined for every $e_t \geq 0$, $\mathcal{V}_t^D$ must be concave, which taking into account that U^{Al} is concave, implies, in turn, that the allocation maximizing the utility of the dynasty head must be symmetric. If, in addition, U^{Al} is strictly concave, the solution to the optimization problem in the definition $\mathcal{V}_t(\overline{e}_0)$ must be unique, and the set of $\mathcal{A}-$efficient allocations reduces to a singleton. Notice that the economy studied by Barro and Becker satisfies these conditions.

S.B.2 Concavity of value functions in an extension of Razin and Ben Zion's (1976) model

But Alvarez's arguments cannot be applied to settings that extend the model of Razin and Ben Zion, in which

$$U^D(x_t, u_{t+1}^D) = u(x_t) + \beta u_{t+1}^D,$$

where u is homothethic.

To be more precise, assume that production functions are time invariant –i.e., $F_t \equiv F$ $\forall t \geq 0$– and costs of rearing children are also time invariant and linear –i.e. $b_t(n_{t+1}) = bn_{t+1}$ $\forall t \geq 0$. In this setting, it is easy to see that each function W_t^D (and, hence, each function π_t) defined in the Supplementary Appendix S.A is both time invariant and separable. Specifically, for each $t \geq 0$ and every $(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D) \in \mathbb{R}_+^2 \times \mathbb{R}$, the function W_t^D can be written as

$$W_t^D(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D) = W(\widehat{e}_t, \widehat{e}_{t+1}) + \beta U_{t+1}^D,$$

where $W : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ is defined, for each $(\widehat{e}_t, \widehat{e}_{t+1}) \in \mathbb{R}^2_+$, by

$$W(\widehat{e}_t, \widehat{e}_{t+1}) = \max_{(k^o_{t+1}, x_t) \in \mathbb{R}^3_+ \times [0,\overline{n}]} \left\{ u(x_t) : c^m_t + b_t(n_{t+1}) + k^o_{t+1} \leq \widehat{e}_t; F(k^o_{t+1}, n_{t+1}) - c^o_{t+1} \geq n_{t+1}\widehat{e}_{t+1} \right\}.$$

More specifically, assume that the utility function u adopts the form

$$u(x_t) = \frac{1}{\theta} \left(v(x_t)\right)^\theta,$$

where $v : \mathbb{R} \to \mathbb{R}$ is non-decreasing, positive valued, concave, differentiable, and linearly homogeneous. Also, write $f(k^m)$ for $f(k^m) = F(k^m, 1)$ and assume that the following holds:

$$\lim_{k^m \longrightarrow 0} \left[(b + k^m) f'(k^m) - f(k^m) \right] > 0. \tag{S.14}$$

With these assumptions, capital *per worker* $k^o(e_t, e_{t+1})/n(e_t, e_{t+1})$ corresponding to a solution $x(e_t, e_{t+1})$ to the optimization problem in the definition of the indirect utility function $W(e_t, e_{t+1})$ depends only on e_{t+1}, that is

$$\frac{k^o(e_t, e_{t+1})}{n(e_t, e_{t+1})} = k^m(e_{t+1}).$$

Also, the indirect utility function adopts the form

$$W(e_t, e_{t+1}) = \frac{1}{\theta} \left(\frac{e_t}{\mathcal{E}(e_{t+1})} \right)^\theta,$$

where $\mathcal{E} : \mathbb{R}_+ \to \mathbb{R}_+$ is the expenditure function defined, for each $e_{t+1} \in \mathbb{R}_+$, by

$$\mathcal{E}(e_{t+1}) = \min_{x_t \in \mathbb{R}^3_+} \left\{ c^m_t + \left(b + \frac{e_{t+1} - w(e_{t+1})}{R(e_{t+1})} \right) n_t + \left(\frac{1}{R(e_{t+1})} \right) c_t \ : \ v(x_t) \geq 1 \right\};$$

and the functions $R(\cdot)$ and $w(\cdot)$ are implicitly defined by

$$\mathcal{E}(e_{t+1}) = e_{t+1};$$

$$R(e_{t+1}) = f'_{t+1}(k^m(e_{t+1}));$$

and

$$w(e_{t+1}) = f(k^m(e_{t+1})) - k^m(e_{t+1}) f(k^m(e_{t+1})).$$

S.B.2.1 Necessary and sufficient conditions for concavity of the value function In the setting described, the value function $\mathcal{V}^D$ is also time invariant and satisfies, for each $e \in \mathbb{R}_+$, by

$$\mathcal{V}^D(e) = \max \left\{ W(e, e') + \overline{\mathcal{V}}^D(e') : e' \in \mathbb{R}_+ \right\}.$$

where, in turn, $\overline{\mathcal{V}}^D(e')$ –defined in (12) in the main paper– is the maximum average utility that the dynasty head obtains by providing her immediate descendants with an average income e', that is,

$$\overline{\mathcal{V}}^D(e') = \max_{E : \mathbb{R}_+ \to [0,1] \in \triangle \mathbb{R}_+} \left\{ \int \mathcal{V}^D(e) dE(e) : \int dE(e) = e' \right\}.$$

Taking this into account, it is straightforward to show that $\mathcal{V}^D$ is concave if W is concave.

In turn, the concavity of the indirect utility function depends on the properties of the expenditure function $\mathcal{E}$ and, to be more precise, on the properties of the function $m : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, defined, for all $e \in \mathbb{R}_+$, by

$$m(e) = \frac{R(e)}{n(e,e)} = \frac{\mathcal{E}(e)}{e\mathcal{E}'(e)}.$$

Thus, m determines the inverse elasticity of current expenditures with respect to future incomes. It is straightforward to show that for any given growth path $\{e_t\}_{t\geq 0}$ we have

$$-\frac{D_1 W(e_t, e_{t+1})}{D_2 W(e_t, e_{t+1})} = \frac{R(e_{t+1})}{n(e_t, e_{t+1})\left(\frac{e_{t+1}}{e_t}\right)} = \left(\frac{e_{t+1}}{e_t}\right) m(e_{t+1}).$$

That is, in an efficient allocation, the ratio of the implicit interest rate to the rate at which total income grows is completely determined by $m(e_{t+1})$. Observe that a non-increasing m is a necessary condition for quasi-concavity of the function W. Also, it is straightforward to show that a sufficient condition ensuring that W is concave is that the function $l : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, defined, for each $e \in \mathbb{R}_+$, by $l(e) = \frac{\mathcal{E}'(e)}{\mathcal{E}'(e)} = em(e)$ is non-increasing.

S.B.2.2 Examples of non-concave indirect utility functions Unfortunately, the function m might be non monotonic. As we show in our previous work (see examples 2 and 3 in Conde-Ruiz *et alli*, 2010) examples of a non-monotonic m can be easily obtained for utility and production functions of the form

$$v(x_t) = \begin{cases} \left[\gamma^m \left(c_t^m\right)^{\frac{\sigma-1}{\sigma}} + \gamma^o \left(c_{t+1}^o\right)^{\frac{\sigma-1}{\sigma}} + \gamma^n (n_{t+1})^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, & \text{if } \sigma \neq 1, \\ \left(c_t^m\right)^{\gamma^m} \left(c_{t+1}^o\right)^{\gamma^o} (n_{t+1})^{\gamma^n}, & \text{if } \sigma = 1, \end{cases}$$

where and $\gamma^m + \gamma^o + \gamma^n = 1$; and

$$F(K,L) = \begin{cases} \left[A\left(K\right)^{\frac{\rho-1}{\rho}} + B\left(L\right)^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}}, & \text{if } \rho \neq 1, \\ \left(K\right)^{\frac{1}{2}} \left(L\right)^{\frac{1}{2}}, & \text{if } \rho = 1. \end{cases}$$

Linear technologies. The simplest family of examples corresponds to economies in which the productions function adopts the linear form $F(k_t^o, n_t) = Rk_t^o + wn_t$, where R and w satisfy $bR > w$. This condition ensures that condition (S.14) is satisfied, which, in turn, implies that $\mathcal{E}(0) > 0$ is satisfied. In this case, the expenditure function $\mathcal{E}(\cdot)$ is concave and the indirect utility function W is quasiconvex. Moreover, the function m decreases if consumption goods are complements, that is, if $\sigma < 0$. For $\sigma > 0$, the function m is *single peak*, that is, there exists a unique point e^* such that m is decreasingly monotonic on $(0, e^*)$ and increasingly monotonic on (e^*, ∞).

Cobb Douglas preferences and non-linear production functions. When preferences adopt a

Cobb-Douglas form corresponding to $\sigma = 1$ we have

$$\begin{aligned} m(e) &= \left(\frac{\gamma^n e}{f'(k^m(e))[b+k^m(e)](\frac{\gamma^n}{\gamma^o+\gamma^n})}\right)^{-1} = \left(\frac{\gamma^n f(k^m(e)) - \gamma^n \frac{\gamma^o[f'(k^m(e)(b+k^m(e))]}{\gamma^o+\gamma^n}}{f'(k^m(e))[b+k^m(e)](\frac{\gamma^n}{\gamma^o+\gamma^n})}\right)^{-1} = \\ &= \left(\frac{(\gamma^o+\gamma^n)\, f(k^m(e))}{f'(k^m(e))[b+k^m(e)]} - \gamma^n\right)^{-1} \end{aligned}$$

In this case, the function $m(\cdot)$ might be increasing or decreasing depending on the properties of the production function. More specifically, for a CES production function we have

$$\frac{f(k^m(e))}{f'(k^m(e))[b+k^m(e)]} = \frac{A\left(k^m(e)\right)^{\frac{\rho-1}{\rho}} + B}{A\left(k^m(e)\right)^{-\frac{1}{\rho}}[b+k^m(e)]} = \frac{k^m(e)}{[b+k^m(e)]} + \frac{B}{A}\frac{\left(k^m(e)\right)^{\frac{1}{\rho}}}{[b+k^m(e)]},$$

which, taking into account that $k^m(\cdot)$ is increasing, implies that $m(\cdot)$ is single peak (as defined above) if inputs are substitutes, i.e. $\rho > 1$.

S.B.2.3 Examples of non-concave value functions The non monotonicity of the function m might also induce non concavities in the unconstrained value function $\mathcal{V}^D$, that, in turn, may introduce differences between the unconstrained value function $\mathcal{V}^D$ and the constrained value function V^D. To see why, suppose m is single peak, that is, suppose there exists e^* such that m is decreasingly monotonic on the interval $[0, e^*)$ and increasingly monotonic on $[0, \infty)$. If $m(e^*) \leq 1$, there are two potential steady states of the dynastic maximization problem, e_s^1 and e_s^2 satisfying first order conditions for dynastic maximization, which, applied to a steady state e_s, reduce to $m(e_s) = \frac{1}{\beta}$. If both e_s^1 and e_s^2 satisfy both second order conditions –given by $D_1W(e_s, e) + \beta D_2W(e, e_s) < 0$ for each $e > e_s$– as well as transversality/dynamic efficiency conditions –given by $\pi(e_s) > 1$– for dynastic maximization. Both steady states e_s^1 and e_s^2 are, respectively, the solutions to the optimization problems in the definition of the –constrained– value functions $V^D(e_s^1)$ and $V^D(e_s^1)$. In this case, we can explicitly compute the constrained values $V^D(e_s^1)$ and $V^D(e_s^1)$ as

$$V^D(e_s^1) = \frac{W(e_s^1, e_s^1)}{1-\beta} \text{ and } V^D(e_s^2) = \frac{W(e_s^2, e_s^2)}{1-\beta}.$$

Suppose now that there exists $\lambda \in (0,1)$ satisfying, for a steady state $e_s \in \{e_s^1, e_s^2\}$

$$W(e_s, \lambda e_s^1 + (1-\lambda)\, e_s^2) + \lambda\beta V^D(e_s^1) + (1-\lambda)\,\beta V^D(e_s^2) > \frac{W(e_s, e_s)}{1-\beta} = V^D(e_s). \quad \text{(S.15)}$$

Observe that the term at the left hand side of (S.15) is the welfare obtained by the dynasty head from her descendants if *a*) total income available to each immediate descendants is e_s; *b*) each immediate descendant provides with e_s^1 units of income to a proportion λ of her immediate descendants and with e_s^2 units of income to a proportion $(1-\lambda)$ of her immediate descendants; and, *c*) each of the two groups of descendants select the symmetric allocations that maximize the utility of the dynasty head among symmetric allocations. Thus, the term at the left hand side of (S.15) is the welfare obtained by the dynasty head with a particular

feasible, non-symmetric allocation that provides the dynasty head with at most the same utility than the allocation that maximizes her utility among feasible allocations. Hence,

$$\mathcal{V}^D(e_s) \geq W(e_s, \lambda e_s^1 + (1-\lambda)\, e_s^2) + \lambda\beta V^D(e_s^1) + (1-\lambda)\,\beta V^D(e_s^2) > V^D(e_s), \qquad \text{(S.16)}$$

therefore establishing that $\mathcal{V}^D$ is not concave in a neighborhood of e_s.

Inequality (S.16) holds for many preferences and production functions for which the function m is single peak –in the sense described above. More precisely, for Cobb-Douglas preferences and a CES technology, if

$$\theta = -0.3; b = 1;\ \beta = 0.5;\ v(x_t) = \left[c_t^m c_{t+1}^o n_t\right]^{\frac{1}{3}};\ \text{and, } F(k_{t+1}^o, n_{t+1}) = \left([k_{t+1}^o]^{\frac{1}{2}} + 3[n_{t+1}]^{\frac{1}{2}}\right)^2,$$

then $e_s^1 = 7.07$, $e_s^2 = 127.33$ and (S.16) holds for $e_s = e_s^2$ and $\lambda > 0.66$. In the case that both preferences and technology are represented by CES function, if $b = 1$; $\beta = 0.71$;

$$\begin{aligned} \theta = -1;\ b = 1;\ \beta = 0.4;\ v(x_t) &= \frac{1}{3^2}\left[(c_t^m)^{\frac{1}{2}} + \left(c_{t+1}^o\right)^{\frac{1}{2}} + (n_t)^{\frac{1}{2}}\right]^2; \quad \text{and,} \\ F(k_{t+1}^o, n_{t+1}) &= \left([k_{t+1}^o]^{\frac{1}{2}} + 3[n_{t+1}]^{\frac{1}{2}}\right)^2; \end{aligned}$$

then $e_s^1 = 4.30$, $e_s^2 = 30.63$ and (S.16) holds for $e_s = e_s^2$ and $\lambda > 0.08$.

The intuition of why non-convexities in the feasible set might generate non-concavities of value functions is the following: when W is not quasiconcave, the function defined by $H(e) = W(e, e)$ is not increasingly monotonic. More precisely, in the examples given above, H reaches a local maximum at the point $e^* > e_s^1$ satisfying $m(e^*) = 1$ and a local minimum at the point $e_* \in (e^*, e_s^2)$, for which $m(e_*) = 1$ is also satisfied. Thus, even though transversality conditions impose that $W(e_s^2, e_s^2) > W(e_s^1, e_s^1)$ is satisfied, the difference $W(e_s^2, e_s^2) - W(e_s^1, e_s^1)$ is small.

Further, it can be shown that, when this occurs, the highest steady state e_s^2 is unstable. That is, the policy function, that is, the function $P : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying, for each $e \in \mathbb{R}_+$, $V^D(e) = W(e, P(e)) + V^D(P(e))$ crosses the line defined by $e_{t+1} = e_t$ from below. Hence, if the dynasty head initial wealth is close to e_s^2, she is almost indifferent between 1) staying at the steady state e_s^2; 2) initiating a sustained growth path for which the income available to each of her descendants is higher and higher and fertility rates are lower and lower; or, 3) initiating a growth path for which the wealth available to each of her descendants is lower and lower and fertility rates are higher and higher, driving the economy towards the steady state e_s^1. Since consumption goods and the number of children are substitutes, the welfare obtained by the dynasty head with either one of these three alternatives is (almost) the same. But, if the dynasty head is not restricted to treat all of her descendants symmetrically, then she can do even better by providing some of their descendants with a relatively low income e_1^s and some of their descendants with a high income e_s^2.

Appendix S.C: Equilibrium behavior

In this Supplementary Appendix S.C, we explore the efficiency properties of a notion of decentralized equilibrium that, as in GJT, results from the combination of the notion of competitive equilibrium and the notion of subgame perfect equilibrium of a voluntary transfer game played within families. Differently from GJT, we assume that *i*) altruism may not be dynastic necessarily; and *ii*) gifts or bequests cannot be negative. As GTJ and other authors exploring transfer games with descendant altruism, such as Leininger (1986), Bernheim and Ray (1989) or Schoonbroodt and Tertilt (2014) we also assume that *iii*) the agents can only observe their own gifts or bequests and those chosen by their parents; and, *iv*) parents cannot condition their gifts and bequests on their children's behavior.[7]

While assumptions *ii*) and *iii*) seem natural in a setting in which the agents are not altruistic towards their ancestors and the agents live, at most, for three periods, *iv*) requires an explanation. After all, parents might introduce legal clauses in their wills that prevent their children from using their bequests in ways that contradict these wills. On the one side, if these clauses were easy to enforce, there would be no need to refine the notion of subgame perfect equilibrium to determine intrafamiliar transfers. As we shall discuss below, in dynastic models, the multiplicity of subgame perfect equilibria arises because the gifts or bequests that any agent would be willing to leave to her immediate descendants depends on the gift or bequest that these descendants decide to leave to their own descendants, and so on. With such clauses, the dynasty head would be able to control her immediate descendants' gifts as well as their own. Since the function U^D determining the utility that the dynasty head obtains from consumption decisions of her descendants has been defined recursively, then any agent's preferences on consumption decisions of her grandchildren coincide with those of her own children. Thus, by controlling the gift behavior of these children, the dynasty head is able to control the sequence of gifts and bequests of all her descendants. Therefore, in the only possible subgame perfect equilibrium arising with such clauses, transfers must maximize the utility of the dynasty head among all possible sequences of (non-negative) gifts or bequests.

On the other side, under Assumptions $A1$ and $A2$ in our paper, parents' preferences and those of their immediate descendants may differ only in the amount of resources that they would be willing to provide to the latter's own children. Therefore, if bequests are imperfectly observable to outside observers, clauses that limit the amount of resources that an agent's children may transfer to the agent's grandchildren will be costly to enforce, mainly because such transfers will benefit the two generations of agents involved affected by such transfers: the agent's children and his/her grandchildren.

In this Supplementary Appendix S.C, we focus on the scenario in which achieving efficiency seems harder, and therefore we do not allow, initially, for this type of clauses. Yet, we shall discuss its effects briefly after a closer look at the equilibrium without clauses is taken.

[7] Golosov, Jones and Tertilt adopt assumption *iv*) in GJT (2007, Sec.4). Assumption *iii*), required, in a general setting, to ensure that the agents' strategies depend only on the wealth inherited from their parents, follows, in their 1-period setting with dynastic altruism, from Subgame Perfection, as explained in GJT (2007, p.1060).

S.C.1 The voluntary transfers game.

Assume that there are two markets operating at each date $t \geq 0$: a financial market, that allows agents to lend (or borrow) an arbitrary amount k^o_{t+1} of the homogeneous good in period t, and obtain (or pay back) a return equal to $R_{t+1}k^o_{t+1}$ units of the same good in period $t+1$; and, a spot job market, in which labor is exchanged against the homogeneous good at a price w_t. We shall assume, without loss of generality, that $\mathcal{D} \equiv \mathcal{D}^O$ holds. Since the agents' preferences may exhibit descendant altruism, each type i^t of an agent of generation t might be willing to transfer, at period $t+1$, an amount $\mathsf{g}_{t+1}(i^t, i_{t+1}) \geq 0$ of the numeraire to each of her immediate descendants when they reach their middle age, which we may refer to as a bequest or a gift depending on whether or not the agents live for one or two periods. By choosing such gifts, parents determine the income scheme $\mathsf{e}_{t+1}(i^t, i_{t+1}) = w_t + \mathsf{g}_{t+1}(i^t, i_{t+1}) \geq w_t$ –and, therefore, the income distribution $E^{\mathsf{e}}_{t+1}(\cdot, i^t)$ available to their descendants. Recall that, for each e, $E^{\mathsf{e}}_{t+1}(e, i^t)$ represents the cumulative probability that a randomly chosen descendant of i^t is endowed with at most e units of the homogeneous good if i^t chooses an income scheme $\mathsf{e}_{t+1}(\cdot)$.

For any allocation a, each period t and each agent $i^t \in \mathbb{R}^t_+$, write $\overline{\mathcal{U}}^D_{t+1}(\mathsf{a}; i^t)$ for the average utility that agent i^t obtains from consumption and fertility decisions of her descendants; that is,

$$\overline{\mathcal{U}}^D_{t+1}(\mathsf{a}; i^t) = \frac{1}{\mathsf{n}_{t+1}(i^t)} \int_{\mathsf{D}_{t+1}(i^t)} \mathcal{U}^D_{t+1}(\mathsf{a}; i^t, i_{t+1}) di_{t+1}.$$

If the agents hold correct expectations both on future prices (represented by a sequence $p^{-t} \equiv \{R_\tau, w_\tau\}_{\tau \geq t+1}$) and on their descendants' future consumption and fertility decisions, then an agent in her middle age at time t, whose income available to finance her consumption, fertility and investment decisions is given by $e_t(i^t) = w_t + g_t(i^t)$, and who provides their immediate descendants with an endowment described by the function $\mathsf{e}_{t+1}(\cdot) \equiv w_{t+1} + \mathsf{g}_{t+1}(\cdot)$, will choose her consumption-fertility bundle $x^*_t(i^t)$ and her savings $k^*_{t+1}(i^t)$ to solve

$$\max_{(k^o_{t+1}, x_t) \in \mathbb{R}^3_+ \times [0, \overline{n}]} \left\{ U\left(x_t, \overline{\mathcal{U}}^D_{t+1}(\mathsf{a}; i^t)\right) : c^m_t + b_t(n_{t+1}) + k^o_{t+1} = \mathsf{e}_t(i^t); \right.$$
$$\left. c^o_{t+1} = R_{t+1}k^o_{t+1} - n_{t+1}\left[\int_{w_{t+1}}^{\infty} e dE^{\mathsf{e}}_{t+1}(e, i^t) - w_{t+1}\right] \right\} \equiv W_{p,t}\left(\mathsf{e}_t(i^t), \int_{w_{t+1}}^{\infty} e dE^{\mathsf{e}}_{t+1}(e, i^t), \overline{\mathcal{U}}^D_{t+1}(\mathsf{a}; i^t)\right).$$

Observe that, by Assumption *A*1 in our paper, any solution $\left(k^*_{t+1}, x^*_t\right)$ to the optimization problem above in the definition of $W_{p,t}(e_t, e_{t+1}, u^D_{t+1})$ is also a solution of the optimization problem in which the objective function is replaced by $U^D\left(x_t, u^D_t\right)$. We shall denote such optimization problem by $W^D_{p,t}(e_t, e_{t+1}, u^D_{t+1})$. Proceeding recursively, it follows that the consumption, investment and fertility decisions taken by an agent of generation t are completely determined by the agent's income $e_t(i^t)$, the sequence of income schemes $\mathsf{e}^{-t} \equiv \left\{\mathsf{e}_{t+\tau} : [0, \overline{n}]^{t+\tau} \to \mathbb{R}_+\right\}_{\tau=1,2,...}$ chosen by the agent and her descendants (which, in turn, is determined by the sequence of gifts schemes $\mathsf{g}^{-t} \equiv \left\{\mathsf{g}_{t+\tau} : [0, \overline{n}]^{t+\tau} \to \mathbb{R}_+\right\}_{\tau=1,2,...}$) and the sequence of prices p^{-t}. Therefore, e and p determine the –indirect– utility payoffs obtained by all agents with a sequence g of voluntary transfers.

Since gifts and prices determine all decisions, including fertility choices, the game of voluntary transfers played within families is closely analogous to those arising in settings with exogenous fertility. As in the classical articles analyzing this type of games in environments with descendant altruism,[8] the assumption imposing that the agents can only observe the gifts chosen by their parents implies that their gift strategies depend only on time[9] –which determines future prices and, hence, the agents' utilty payoffs– and on their income available after their parents have chosen their gifts. If strategies are symmetric,[10] that is, if any two agents of the same generation receiving the same gift choose the same gift scheme for their immediate descendants, then the income distribution arising with a symmetric strategy of an agent of generation t can be represented by a mapping[11]

$$E^{\mathsf{e}}_{t+1} : \mathbb{R}_+ \to \triangle [w_{t+1}, \infty)$$

that determines, for any $e_t \geq w_t$, the distribution of income $E_{t+1}(\cdot/e_t)$ that results from the gift strategy of an agent of generation t. Note that, as in a dynastic optimum, the support of the income distribution $E_{t+1}(\cdot/e_t)$ need not be a singleton. The utility payoffs obtained by an arbitrary agent of generation t who receives an income e_t with the sequence of strategies $E^{-t} = \{E_{t+\tau} : \mathbb{R}_+ \to \triangle [w_{t+1}, \infty)\}_{\tau=1,2,...}$ can be written, for $t \geq 0$, as

$$\pi_{p,t}(e_t, E^{-t}) = W_{p,t}\left(e_t, \int_{w_{t+1}}^{\infty} e dE_{t+1}(e/e_t), \int_{w_{t+1}}^{\infty} \pi^D_{p,t+1}(e, E^{-(t+1)}) dE_{t+1}(e/e_t)\right);$$

where $e_0 = \overline{e}_0$ and, in turn, $\pi^D_{p,t}(e_t, E^{-t})$ is recursively defined, for $t \geq 1$ and $e_t \geq w_t$, by

$$\pi^D_{p,t}(e_t, E^{-t}) = W^D_{p,t}\left(e_t, \int_{w_{t+1}}^{\infty} e dE_{t+1}(e/e_t), \int_{w_{t+1}}^{\infty} \pi^D_{p,t+1}(e, E^{-(t+1)}) dE_{t+1}(e/e_t)\right).$$

As we shall see below, there might exist, in general, many symmetric strategies arising as a Subgame Perfect Equilibria (hereafter, SPE) of the voluntary transfers game played within families. Among these, we select the sequence of strategies that maximizes the utility of the dynasty head among all possible SPE –with symmetric strategies– of the game. In this notion of equilibrium, the agents proceed as if their preferences were purely recursive and $U^D \equiv U$ was satisfied, so that, when altruism is dynastic, our notion of equilibrium coincides with that explored by Schoonbroodt and Tertilt (2014). Furthermore, if the non-negativity constraints on bequests is not binding, our notion of equilibrium gives rise to the same allocation than the one explored in the classical papers by Razin and Ben Zion (1976), Barro and Becker (1989), or Golosov, Jones and Tertilt (2007, Sec.4) themselves, who focus

[8] See, e.g., Leininger (1986).

[9] In some of these papers, for example, Berbheim and Ray (1989), prices are assumed to be constant or to follow a stationary random process, and therefore the gift strategies selected by the agents are Markov strategies depending only on the gift –or income– received by the agents and –possibly– a time-independent random shock.

[10] As it is standard in Game Theory, that the agents play a symmetric strategy means that all agents with the same preferences and strategic possibilities choose the same strategy. Due to non-convexities, the symmetric strategies corresponding to an equilibrium give rise to *ex-ante* symmetric allocations that may not be symmetric *ex-post*.

[11] As in the main paper, $\triangle [w_{t+1}, \infty)$ represents the set of distribution functions with support on the set $[w_{t+1}, \infty)$.

on a setting with dynastic altruism in which the individuals live for one period and the non-negativity constraints on bequests cannot be binding. In absence of altruism, our notion of equilibrium coincides with the "*laissez faire*" equilibrium studied by Conde-Ruiz *et alii* (2010, Sec.4).

S.C.2 Equilibrium in the voluntary transfers game.

To define our notion of equilibrium in a more precise way, consider what the payoffs of the game would be like under the assumption that altruism is of the dynastic type, so that $U^D \equiv U$ holds. With this assumption, the (indirect) utility payoffs $\pi^*_{p,t}(e_t, E^{-t})$ that the dynasty head would obtain with a sequence E from consumption decisions of her descendants of generation t coincides with the utility obtained by these descendants and can be recursively defined, for $t \geq 0$, by

$$\pi^*_{p,t}(e_t, E^{-t}) = W_{p,t}\left(e_t, \int_{w_{t+1}}^{\infty} e dE_{t+1}(e/e_t), \int_{w_{t+1}}^{\infty} \pi^*_{p,t+1}(e, E^{-(t+1)}) dE_{t+1}(e/e_t)\right);$$

with $e_0 = \overline{e}_0$. Thus, under the assumption that $U \equiv U^D$ holds, the dynasty head will select a sequence E^* of income schemes such that, for each $t \geq 0$ and each $e_t \geq w_t$, the sequence E^{*-t} solves

$$\mathcal{V}^*_{p,t}(e_t) = \max_{E^{-t}} \pi^*_{p,t}(e_t, E^{-t}), \tag{S.17}$$

so that the sequence of value functions $\left\{\mathcal{V}^*_{p,t}\right\}_{t\geq 0}$ satisfies, for each $t \geq 0$ and each $e_t \geq w_t$,

$$\mathcal{V}^*_{p,t}(e_t) = \max_{E:[w_t,\infty)\to[0,1]\in\Delta[w_t,\infty)} W_{p,t}\left(e_t, \int_{w_{t+1}}^{\infty} e dE(e), \int_{w_{t+1}}^{\infty} \mathcal{V}^*_{p,t+1}(e) dE(e)\right). \tag{S.18}$$

It is straightforward to show that, with dynastic altruism, the sequence E^* forms a SPE of the game and maximizes the utility of the dynasty head among all possible SPE in which the agents play Markov strategies. To see this, suppose there exists a period t in which the dynasty head obtains utility gains if some of her descendants –for example, those receiving an income e_t– deviate from the sequence E^* by choosing an income scheme $E_{t+1} \neq E^*_{t+1}$. That is, suppose there exists $t \geq 0$, a number e_t in the support of the E^*_t and a transfer scheme $E_{t+1} \neq E^*_{t+1}$ for which

$$W^D_{p,t}\left(e_t, \int_{w_{t+1}}^{\infty} e dE_{t+1}(e), \int_{w_{t+1}}^{\infty} \mathcal{V}^*_{p,t+1}(e) dE(e)\right) > W^D_{p,t}\left(e_t, \int_{w_{t+1}}^{\infty} e dE^*_{t+1}(e), \int_{w_{t+1}}^{\infty} \mathcal{V}^*_{p,t+1}(e) dE(e)\right),$$

is satisfied. By Assumption $A2$ we have

$$W_{p,t}\left(e_t, \int_{w_{t+1}}^{\infty} e dE_{t+1}(e), \int_{w_{t+1}}^{\infty} \mathcal{V}^*_{p,t+1}(e) dE(e)\right) > W_{p,t}\left(e_t, \int_{w_{t+1}}^{\infty} e dE^*_{t+1}(e), \int_{w_{t+1}}^{\infty} \mathcal{V}^*_{p,t+1}(e) dE(e)\right),$$

which implies that such descendant is not playing a best response to their descendants' strategies and, hence, contradicts the hypothesis stating that E^* is a subgame perfect equilibrium. Thus, Assumption $A2$ ensures that, even in environments in which $U \equiv U^D$ does not hold, E^* is also a SPE of the game and maximizes the utility of the dynasty head among all possible SPE in which the agents play symmetric strategies.

We should also point out that the sequence of functional equations in (S.18) characterizes the sequence of payoffs that would correspond to a sequence of SPE strategies if the model were truly dynastic and $U^D = U$ was satisfied. To select among them, GTJ[12] have shown that only the sequence of value functions $\left\{\mathcal{V}^*_{p,t}\right\}_{t\geq 0}$, as defined in (S.17), corresponds to the sequence of equilibrium payoffs arising as the limit, as T tends to infinity, of the sequence of equilibrium payoffs arising in a truncated version of the game that lasts $T+1$ periods and provides all agents living after the last of this periods with an income[13] $e_t = w_t$ for $t > T+1$. This result suggests that the sequence $\left\{\mathcal{V}^*_{p,t}\right\}_{t\geq 0}$ is the one corresponding to the limit of the equilibrium payoffs of a sequence of truncated games even when altruism is non-dynastic. Yet, to save on space, we will not prove this claim.

S.C.3 Decentralized Equilibrium

With this equilibrium selection criterion for the game played within families, the interaction of markets and families gives rise to an allocation, a sequence of income schemes and a sequence of prices that we shall refer to as a decentralized equilibrium. A decentralized equilibrium, therefore, can be defined formally as follows:

Definition 2 *A decentralized equilibrium is a feasible allocation* a^**, a sequence of income schemes* $\mathsf{e}^* = \left\{\mathsf{e}^*_{t+1} : \mathbb{R}_+^{t+1} \longrightarrow \mathbb{R}_+\right\}_{t\geq 0}$ *and a sequence of prices* $p \equiv \{R_t, w_t\}_{t\geq 1}$ *such that, for each* $t \geq 0$*,*
i) aggregate capital (K^*_{t+1}) *and labor* (L^*_{t+1}) *chosen by firms maximize profits, that is,*

$$D_1 F_{t+1}(K^*_{t+1}, L^*_{t+1}) = R_{t+1} \text{ and } D_2 F_{t+1}(K^*_{t+1}, L^*_{t+1}) = w_{t+1}$$

are satisfied;
ii) given e^**, each agent* $i^t \in \mathsf{D}^t(i^0)$ *chooses* $\mathsf{x}^*_t(i^t)$ *to maximize utility; that is,*

$$\mathcal{U}_t(\mathsf{a}^*; i^t) = W_{p,t}\left(\mathsf{e}^*_t(i^t), \int_{w_{t+1}}^{\infty} e dE^{\mathsf{e}^*}_{t+1}(e, i^t), \overline{\mathcal{U}}^D_{t+1}(\mathsf{a}^*; i^t)\right);$$

iii) capital and labor markets clear, that is,

$$K^*_{t+1} = \int_{i^t \in \mathsf{D}^t(i^0)} \mathsf{k}^{o*}_{t+1}(i^t) di^t \text{ and } L^*_{t+1} = \int_{i^t \in \mathsf{D}^t(i^0)} \mathsf{n}^*_{t+1}(i^t) di^t$$

are satisfied; and,
iv) e^* *is the outcome of the SPE of the voluntary transfers game that maximizes the utility of the dynasty head among all possible subgame perfect equilibria (corresponding to symmetric strategies) of the voluntary transfers game played within families; that is, for all* t*, and all* $i^t \in [0, \overline{n}]^t$*, the sequence* $E^{\mathsf{e}*}_{t+1}(\cdot, i)$ *solves*

$$\max_{E:[w_t,\infty)\to[0,1]\in\Delta[w_t,\infty)} W_{p,t}\left(e_t(i^t), \int_{\mathbb{R}_+} e dE(e), \int_{\mathbb{R}_+} \mathcal{V}^*_{p,t+1}(e) dE(e)\right).$$

[12] The original argument corresponds to Fudemberg and Levine (1993).
[13] Since GJT do not take into account non-negativity constraint on gifts, they assume $w_t = 0$.

S.C.4 Ex-post symmetric equilibria. A Welfare Theorem.

When each value function $\mathcal{V}^*_{p,t}$ is strictly concave on $[w_t, \infty)$, the allocation a^* corresponding to a decentralized equilibrium must be (ex-post) symmetric, that is, all agents of the same generation take the same consumption and fertility decisions. We should point out that conditions ensuring the concavity of each value function $\mathcal{V}^*_{p,t}$ are analogous than those ensuring the concavity of each value function $\mathcal{V}^D_t$ in settings with infinite horizon altruism, although, in order to ensure the symmetry of equilibria, it suffices that each value function $\mathcal{V}^*_{p,t}$ is concave on the interval $[w_t, \infty)$. Therefore, even if we allow for random strategies, the interaction of markets and families in the framework analyzed in the paper delivers, under relatively weak conditions, an *ex-post* symmetric allocation.

Theorem S.C.1 below shows that such symmetric equilibria are (statically) Millian efficient and, if Property $\boldsymbol{S}$ holds[14] and the equilibrium is dynamically efficient, then it is both Millian efficient and $\mathcal{P}-$efficient.

Theorem S.C.1 *Let $\widehat{a}$ be an allocation corresponding to a symmetric competitive equilibrium.*

i) $\widehat{a}$ is statically Millian efficient.

ii) If $\widehat{a}$ is dynamically Millian efficient and that the sequence $\{\mathcal{U}^N_t\}_{t\geq 1}$ satisfies Property $\boldsymbol{S}$, then $\widehat{a}$ is also $\mathcal{P}-$efficient.

Proof of Theorem S.C.1. To prove Theorem S.C.1, let $\widehat{a}$ be a symmetric, decentralized equilibrium associated to a sequence of prices p. By letting $w^*_{p,t}(e_t, e^{-t})$ be recursively defined, for each t, by $w^*_{p,t}(e_t, e^{-t}) = W_{p,t}(e_t, e_{t+1}, w^*_{p,t}(e_{t+1}, e^{-(t+1)}))$, it is straightforward to show that sequence $\{\widehat{e}_t\}_{t\geq 1}$ corresponding to a decentralized equilibrium solves, for each $t \geq 0$ the optimization problem

$$v^*_{p,t}(\widehat{e}_t, \widehat{e}^{-t}) = \max\left\{w^*_{p,t}(\widehat{e}_t, e^{-t}) : e^{-t} \geq \widehat{e}^{-t}\right\}.$$

By Assumption *A*2, the sequence $\{\widehat{e}_t\}_{t\geq 1}$ also solves

$$v^D_{p,t}(\widehat{e}_t, \widehat{e}^{-t}) = \max\left\{w^D_{p,t}(\widehat{e}_t, e^{-t}) : e^{-t} \geq \widehat{e}^{-t}\right\};$$

where $w^D_{p,t}(e_t, e^{-t})$ is recursively defined, for each t, by

$$w^D_{p,t}(e_t, e^{-t}) = W^D_{p,t}(e_t, e_{t+1}, w^D_{p,t+1}(e_{t+1}, e^{-(t+1)})).$$

Taking this into account, we now show that one must have, for $t \geq 1$,

$$w^D_t(\widehat{e}_t, \widehat{e}^{-t}) = v^D_t(\widehat{e}_t, \widehat{e}^{-t}) = \max\left\{w^D_t(\widehat{e}_t, e^{-t}) : e^{-t} \geq \widehat{e}^{-t}\right\}. \tag{S.19}$$

[14]The Property $\boldsymbol{S}$ is motivated and formally presented in Section 5 in our paper:
Property *S*. *For every $t \geq 1$, every $i^t \in [0, \overline{n}]$ and every ex-post symmetric allocation a such that $\mathsf{x}_t(i^t) = x_t$ and $\mathsf{n}_{t+1}(i^t) = n_{t+1}$ one has $\mathcal{U}^N_t(\mathsf{a}; i^t) = \mathcal{U}_t(\mathsf{a}; i^t) \equiv U_t(a)$.*

To prove (S.19) is satisfied, suppose it is not. That is, there exists $\{\widetilde{e}_t\}_{t\geq 1}$ and $\tau \geq 1$ for which $\widetilde{e}^{-\tau} \geq \widehat{e}^{-\tau}$ and $w_\tau^D(\widetilde{e}_\tau, \widetilde{e}^{-\tau}) \geq w_\tau^D(\widehat{e}_\tau, \widehat{e}^{-\tau})$ is satisfied. Also, since in a competitive equilibrium firms maximize profits one must have

$$F_{t+1}(\widehat{k}_{t+1}, \widehat{n}_{t+1}) - R_{t+1}\widehat{k}_{t+1} - w_{t+1}\widehat{n}_{t+1} = 0 \geq F_{t+1}(\widetilde{k}^o_{t+1}, \widetilde{n}_{t+1}) - R_{t+1}\widetilde{k}^o_{t+1} - w_{t+1}\widetilde{n}_{t+1}.$$

Then, $F_{t+1}(\widetilde{k}^o_{t+1}, \widetilde{n}_{t+1}) - \widetilde{n}_{t+1}\widetilde{e}_{t+1} \leq R_{t+1}\widetilde{k}^o_{t+1} + w_{t+1}\widetilde{n}_{t+1}$ and, therefore, the sequence $\{(\widehat{x}^p_\tau, \widehat{k}^p_\tau)\}_{\tau=t}^\infty$ that solves the sequence of optimization problems in the definition of $w_t^D(\widetilde{e}_\tau, \widetilde{e}^{-t})$ is feasible in the sequence of optimization problems in the definition of $w_{p,t}^D(\widetilde{e}_t, \widetilde{e}^{-t})$. This yields,

$$w_{p,t}^D(\widehat{e}_t, \widehat{e}^{-t}) = w_t^D(\widehat{e}_t, \widehat{e}^{-t}) \geq w_{p,t}^D(\widetilde{e}_t, \widetilde{e}^{-t}) \geq w_t^D(\widetilde{e}_t, \widetilde{e}^{-t}),$$

for $t \geq 1$, a contradiction that establishes (S.19). By Proposition S.2 in Pérez-Nievas *et alli* (2017, Supplementary Appendix S.A), the allocation $\widehat{a}$ is statically efficient.

To complete the proof, use again the profit maximizing conditions to show that a solution of the maximization problem $\mathcal{V}_{p,t}(\widehat{e}_t)$ is also a solution of the maximization problem $\mathcal{V}_{\widehat{e},t}(\widehat{e}_t)$, hence $\mathcal{V}_{p,t}(\widehat{e}_t) = \mathcal{V}_{\widehat{e},t}(\widehat{e}_t)$. Proceeding as in the proof of Theorem 2, this establishes that the allocation $\widehat{a}$ corresponding to a competitive equilibrium is $\mathcal{P}-$efficient, which completes the proof. ■

Thus, for all the altruism and utility specifications considered in our paper, a version of the First Welfare Theorem holds for Millian efficiency and, hence, for $\mathcal{P}-$efficiency. Therefore, when applied to Millian efficiency or $\mathcal{P}-$efficiency, potential market failures are of the same nature as those affecting Pareto efficient allocations in dynamic economies with exogenous fertility: although competitive equilibria are always statically efficient, they might be inefficient (or dynamically inefficient). From our characterization of Millian efficient allocations (see Proposition S.A.3), it is straightforward to see that, for the allocation $\widehat{a}$ corresponding to the competitive equilibrium, a sufficient condition for dynamic Millian efficiency is that

$$\lim_{T\to\infty}\left(b'_T(0) - \frac{\widehat{w}_{T+1}}{\widehat{R}_{T+1}}\right)\frac{\widehat{n}_{T+1}}{\widehat{e}_T} > 0$$

is satisfied. That is, a sufficient condition ensuring the Millian efficiency (and, hence, the $\mathcal{P}-$efficiency) long-run wages do not exceed the capitalized costs of rearing children.

S.C.5 Discussion.

We would like to emphasize that the $\mathcal{P}-$efficiency of decentralized equilibrium holds under weaker conditions than those required to establish their $\mathcal{A}-$efficiency. Note that, even if they are not *ex-post* symmetric, equilibria are *ex-ante* symmetric. Therefore, in view of Corollary 1 in our paper, and even if we identify potential agents by their positions in their siblings' birth order, an equilibrium may be $\mathcal{A}-$efficient only if it is a dynastic optimum. Therefore, an equilibrium arising with non-dynastic altruism, or that for which the non-negativity constraint on transfers is binding, cannot be $\mathcal{A}-$efficient.

The possibility that competitive equilibria arising with voluntary transfers are $\mathcal{A}-$inefficient when the non-negativity constraint is binding has also been noticed before by Schoon-

broodt and Tertilt (2014),[15] who view this possibility as a market failure –or, in the authors' words, "an instance in which Coase Theorem does not apply" (p.566), that arises because parents do not have rights on their children's labor income– and suggest to correct it with fertility dependant pension schemes. In our view, we should take this claim on the failure of Coase theorem (and the first Welfare Theorem), as well as the policy proposals that accompany it, with caution. First of all, the fact that the efficiency in the allocation of resources depends on the initial allocation of property rights means that the so called "Coase Theorem" does not hold... when applied to $\mathcal{A}-$efficiency. But, in a standard interpretation of Coase Theorem,[16] Pareto efficiency arises from any distribution of rights –in absence of transaction costs– because Pareto inefficiency means "unexploited gains from trade", and it is not clear to us that $\mathcal{A}-$inefficiency means the same thing. Perhaps it would if 1) the agents could really trade in a market for private contracts between parents and their children in which the latter commit to compensate the former for child expenses; and 2) true preferences of potential unborn people were such that, for any allocation of resources, any potential agent would prefer to be alive rather than not being born. But this is precisely the problem: since trade on "the right to be born" is impossible, we cannot obtain the information needed to know not only whether an allocation is efficient or not, but also to know whether or not 2) holds and, consequently, to know what efficiency means.

Second of all, in our setting, the symmetric equilibria with strictly positive transfers from parents to children arising in a setting with dynastic altruism cannot be distinguished, from the point of view of an outside observer, from those arising with non-dynastic altruism. However, the former are $\mathcal{A}-$efficient, while the latter are not. Moreover, in the latter case, fertility dependant pension schemes might be ineffective as a means to achieve $\mathcal{A}-$efficiency, because, in this case, the inefficiency of markets arises because parents cannot introduce clauses in their wills that force their children not to leave any bequests to their grandchildren. However, allowing and enforcing such provisions in the agents' wills would involve considerable transaction costs, and (in case altruism extends only to a finite number of periods) would drive the economy to a collapse.

In contrast with $\mathcal{A}-$efficiency, the $\mathcal{P}-$efficiency of decentralized equilibria is robust to variations on several of the assumptions (on the *ex-post* symmetry of equilibria, on the impossibility of introduce clauses in the agents' wills, on the criterion to select among the many equilibria of the dynastic game and on the possibility that the agents can observe and recall past moves in the game) made to establish Theorem S.C.1. To be more precise:

Non-symmetric equilibria. Although Theorem S.C.1 applies only to ex-post symmetric equilibria, it does not state than an equilibrium that is not ex-post symmetric is $\mathcal{P}-$inefficient. To see this, suppose that the utility attributed to the unborn adopts the form given in (23). Observe that, for any equilibrium path (which is always ex-ante symmetric) for which non-negativity constraints are not binding, the sequence of income schemes corresponding to such equilibrium maximizes the utility that the dynasty head can obtain

[15] This paper explores, in a setting with dynastic altruism, whether or not the First Welfare Theorem holds for the equilibria arising if parents can obtain from their children only a given fraction (possibly, zero) of their income.

[16] See Mas-Colell *et alli* (1995, p.350).

among all trajectories for which all their descendants also choose their income schemes to maximize their own utility. Thus, if the dynasty head chooses another trajectory of income schemes, some of her descendants will be worse off. The dynasty head can only be better off by having more children and providing them with lower resources than the resources obtained by their living siblings, which is not a welfare improvement from the point of view of the $\mathcal{P}-$dominance criterion. Thus, an equilibrium for which the non-negativity constraints on transfers are never binding is $\mathcal{P}-$efficient, even if altruism is non-dynastic and the equilibrium is not ex-post symmetric.

Unfortunately, if the non-negativity constraints on transfers are binding, things are subtler. It is still true that the dynasty head maximizes the utility that she can obtain by providing her immediate descendants with at least w_t units of the homogeneous good, provided all her descendants will also decide their gifts to maximize their own utility (also with a non-negativity constraint on transfers). Yet, in this case, this fact does not ensure the $\mathcal{P}-$efficiency of an equilibrium. By the same reasons by which an ex-post symmetric equilibrium might fail to be dynamically efficient, an equilibrium for which some of the constraints on transfers are binding might not be $\mathcal{P}-$efficient, and it might be improved upon by a sequence of transfers from all agents to their parents. However, identifying sufficient conditions for dynamic efficiency of allocations that are not ex-post symmetric is out of the scope of our paper.

Equilibria with clauses in the agents' wills. The $\mathcal{P}-$efficiency of equilibria also holds if we abandon the assumption imposing that the agents cannot impose clauses in their wills that prevent their descendants from using their gifts or bequests in certain ways. If such clauses are allowed, the dynasty head will select a sequence E^* of income schemes such that, for each $t \geq 1$ and each $e_t \geq w_t$, the sequence E^{*-t} solves

$$\mathcal{V}_{p,t}^D(e_t) = \max_{E^{-t}} \pi_{p,t}^D(e_t, E^{-t}) = \max_{E^{-t}} \left\{ W_{p,t}^D \left(e_t, \int_{w_{t+1}}^{\infty} e dE_{_{t+1}}(e/e_t), \int_{w_{t+1}}^{\infty} \pi_{p,t+1}^D(e, E^{-(t+1)}) dE_{_{t+1}}(e/e_t) \right) \right\},$$

so that the sequence of value functions $\left\{\mathcal{V}_{p,t}^D\right\}_{t\geq 0}$ satisfies, for each $t \geq 0$ and each $e_t \geq w_t$,

$$\mathcal{V}_{p,t}^D(e_t) = \max_{E:[w_t,\infty)\to[0,1]\in\Delta[w_t,\infty)} W_{p,t} \left(e_t, \int_{w_{t+1}}^{\infty} e dE(e), \int_{w_{t+1}}^{\infty} \mathcal{V}_{p,t+1}^D(e) dE(e) \right).$$

As for the case in which clauses are not allowed, the equilibrium of the dynastic game is symmetric (under certain conditions). Taking this into account, it is straightforward to proceed as in the proof of Theorem S.C.1 to show that a symmetric equilibrium of the dynastic game (with clauses) is also $\mathcal{P}-$efficient. In this case, it may also be $\mathcal{A}-$efficient even if altruism is of the non-dynastic type. Yet, achieving $\mathcal{A}-$efficiency requires that the non-negativity constraints on transfers are never binding.

Other Subgame Perfect Equilibria. The $\mathcal{P}-$efficiency of equilibria might hold also if we consider other subgame perfect equilibria of the dynastic game. If we keep the assumption that the agents play symmetric strategies, any solution $\{\mathcal{V}_{p,t} : \mathbb{R}_+ \to \mathbb{R}_+\}_{t\geq 0}$ to a sequence

of functional equations of the form

$$\mathcal{V}_{p,t}(e_t) = \max_{E:\mathbb{R}_+\to[0,1]\in\Delta[w_t,\infty)} W_{p,t}\left(e_t, \int_{w_{t+1}}^{\infty} e dE(e), \int_{w_{t+1}}^{\infty} \mathcal{V}_{p,t+1}(e) dE(e)\right).$$

determines the sequence of utility payoffs corresponding to a subgame perfect equilibrium. If an equilibrium is ex-post symmetric, then the sequence of functional equations characterizing equilibria can be written as

$$\mathcal{V}_{p,t}(e_t) = \max_{e_{t+1}\in\mathbb{R}_+} \left\{W_{p,t}\left(e_t, e_{t+1}, \mathcal{V}_{pt+1}(e_{t+1})\right) : e_{t+1} \geq w_{t+1}\right\}.$$

Taking this into account, and proceeding as in Theorem S.C.1, it is straightforward to show that any (ex-post) symmetric, subgame perfect equilibrium of the dynastic game must be statically Millian efficient. If in addition, the allocation corresponding to such equilibrium is dynamically efficient, then it is also $\mathcal{P}-$efficient. Furthermore, none of these equilibria –except the equilibrium generating the sequence of payoffs functions $\left\{\mathcal{V}^*_{p,t} : \mathbb{R}_+ \to \mathbb{R}_+\right\}_{t\geq 0}$ defined above– can be $\mathcal{A}-$efficient, because the sequence $\left\{\mathcal{V}^*_{p,t} : \mathbb{R}_+ \to \mathbb{R}_+\right\}_{t\geq 0}$ identifies the sequence of equilibrium payoffs corresponding to the equilibrium that maximizes the utility of the dynasty head among all subgame perfect equilibria.

The game with perfect recall. Finally, if we abandon the assumption imposing that the agents can only observe their own gifts or bequests and those chosen by their parents, other equilibria may emerge, as it occurs in many games with overlapping generations games with non altruistic players and exogenous fertility (see, e.g., Lones Smith 1992). Some of them –that involve transfers from the middle-aged agents to the old agents– may even restore (Pareto) efficiency when equilibrium outcomes are dynamically inefficient, as shown by Hammond (1975). We should note, however, that –apart from the rather demanding assumptions on the capacity of the agents to recall information on past moves– in this type of folk theorems, an equilibrium differing from the equilibria considered here may emerge if it provides the agents with more utility than the utility they obtain with the "disagreement point." Taking this into account, suppose that the decentralized equilibrium, as defined above, is symmetric and dynamically Millian efficient, but fails to be $\mathcal{A}-$efficient. Will the agents coordinate in an equilibrium that would make all the agents –except the dynasty head– worse off? They will certainly not if the non-negativity constraints on gifts or bequests are binding. If we take the allocation for which $e_t = w_t$ for $t \geq 1$ as the disagreement point, achieving $\mathcal{A}-$efficiency requires, in this case, that the middle-aged agents transfer resources to the old agents; that is, it requires that some of the agents are worse off than they are with the disagreement point.

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